## CHAPTER 17 - FOURIER TRANSFORM

List of topics for this chapter :

- Fourier Transform and its Properties
- Circuit Applications
- Parseval's Theorem
- Applications

### FOURIER TRANSFORM AND ITS PROPERTIES

**Problem 17.1** Find the Fourier Transform of the pulse shown in Figure 17.1.

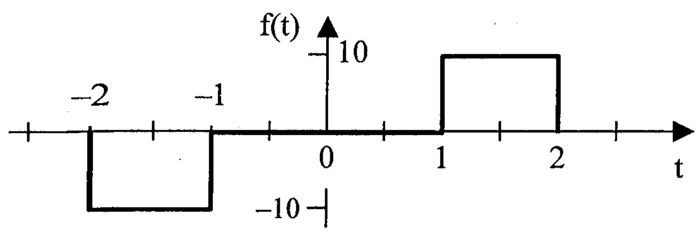


**Figure 17.1**

We begin with the derivative of f(t).

$$\frac{df(t)}{dt} = -\delta(t+2) + \delta(t+1) + \delta(t-1) - \delta(t-2)$$

Transforming this into the frequency domain yields,

$$j\omega F(\omega) = -e^{j2\omega} + e^{j\omega} + e^{-j\omega} - e^{-j2\omega} = 2\cos(2\omega) - 2\cos(\omega)$$

Therefore,

$$F(\omega) = \frac{\mathbf{2(\cos(2\omega) - \cos(\omega))}}{\mathbf{j\omega}}$$

**Problem 17.2** Find the inverse Fourier transforms of the following,

(a) $10/[(j\omega)(j\omega + 5)]$
(b) $5j\omega/[(-j\omega + 1)(j\omega + 2)]$
(c) $(2 - j\omega)/(-\omega^2 + 4j\omega + 3)$
(d) $3\delta(\omega)/[(j\omega + 2)(j\omega + 3)]$

Now to find the inverse transforms.

(a) $F(s) = \dfrac{10}{s(s+5)} = \dfrac{A}{s} + \dfrac{B}{s+5}$, $A = 10/5 = 2$ and $B = 10/-5 = -2$

Therefore, $F(\omega) = (2/j\omega) - (2/(j\omega + 5))$

Transforming, $f(t) = \underline{\mathbf{sgn(t) - 2e^{-5t}u(t)}}$

(b) $F(s) = \dfrac{5s}{(-s+1)(s+2)} = \dfrac{-5s}{(s-1)(s+2)} = \dfrac{A}{s-1} + \dfrac{B}{s+2}$

$A = -5/(1+2) = -5/3$ and $B = -5x(-2)/(-2-1) = -10/3$

Therefore, $F(\omega) = [(-5/3)/(j\omega - 1)] + [(-10/3)/(j\omega + 2)]$

Transforming, $f(t) = \underline{\mathbf{-\frac{5}{3}e^{t}u(-t) - \frac{10}{3}e^{-2t}u(t)}}$

(c) $F(s) = \dfrac{(2-s)}{(s+1)(s+3)} = \dfrac{A}{s+1} + \dfrac{B}{s+3}$, $A = 3/2$ and $B = -5/2$

Therefore, $F(\omega) = [1.5/(j\omega + 1)] - [2.5/(j\omega + 3)]$

Transforming, $f(t) = \underline{\mathbf{1.5e^{-t}u(t) - 2.5e^{-3t}u(t)}}$

(d) $f(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \dfrac{3\delta(\omega)e^{j\omega t}}{(j\omega+2)(j\omega+3)}dt = \dfrac{1}{2\pi}\dfrac{3}{6} = \underline{\mathbf{\dfrac{1}{4\pi}}}$

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**Problem 17.3** *[17.7]* Find the Fourier transform of the "sine-wave pulse" shown in Figure 17.2.

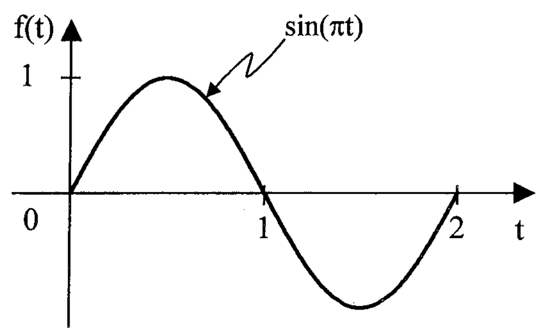



**Figure 17.2**

$f(t) = \sin(\pi t)[u(t) - u(t-2)]$

$$F(\omega) = \int_0^2 \sin(\pi t)\, e^{-j\omega t}\, dt = \frac{1}{2j}\int_0^2 (e^{j\pi} - e^{-j\pi})(e^{-j\omega t})\, dt$$

$$F(\omega) = \frac{1}{2j}\int_0^2 e^{j(-\omega+\pi)t} + e^{-j(\omega+\pi)t}\, dt$$

$$F(\omega) = \frac{1}{2j}\left[\frac{1}{-j(\omega-\pi)} e^{-j(\omega-\pi)t}\Big|_0^2 + \frac{1}{-j(\omega+\pi)} e^{-j(\omega+\pi)t}\Big|_0^2\right]$$

$$F(\omega) = \frac{1}{2}\left(\frac{1-e^{-j2\omega}}{\pi-\omega} + \frac{1-e^{-j2\omega}}{\pi+\omega}\right)$$

$$F(\omega) = \frac{1}{(2)(\pi^2-\omega^2)}(2\pi + 2\pi e^{-j2\omega})$$

$$F(\omega) = \underline{\frac{\pi}{\omega^2-\pi^2}(e^{-j2\omega} - 1)}$$

**CIRCUIT APPLICATIONS**

**Problem 17.4** Find the transfer function, $V_o(\omega)/V_s(\omega)$ for the circuit shown in Figure 17.3.

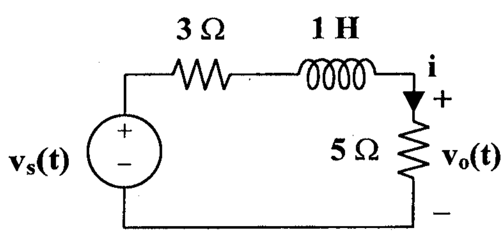



**Figure 17.3**

First we will solve for I.

$$I = \frac{V_s(\omega)}{3 + j\omega + 5} = \frac{V_s(\omega)}{j\omega + 8}, \text{ and } V_o(\omega) = 5I$$

Therefore,
$$\frac{V_o(\omega)}{V_s(\omega)} = \underline{\frac{5}{j\omega + 8}}$$

**Problem 17.5** Solve for $v_C(t)$ in Figure 17.4, where $i(t) = u(t)$ A.

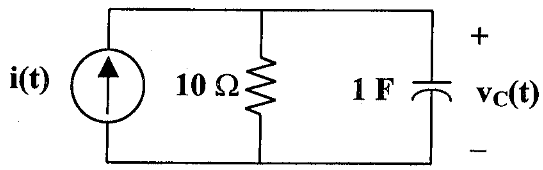



**Figure 17.4**

First we transform i(t) into the frequency domain.

$$I(\omega) = \pi\delta(\omega) + 1/(j\omega), \text{ and } V_C(\omega) = I(\omega)\frac{10\frac{1}{j\omega}}{10+\frac{1}{j\omega}} = I(\omega)\frac{1}{j\omega+0.1}$$

Therefore, $V_C(\omega) = \dfrac{\pi\delta(\omega)}{j\omega} + \dfrac{1}{j\omega(j\omega+0.1)} = V_1 + V_2$

$V_2 = \dfrac{1}{s(s+0.1)} = \dfrac{A}{s} + \dfrac{B}{s+0.1}$, where $A = 1/0.1 = 10$ and $B = 1/(-0.1) = -10$

Therefore, $v_2(t) = 5\text{sgn}(t) - 10e^{-t/10}u(t)$

$$v_1(t) = \frac{1}{2\pi}\int \frac{\pi\delta(\omega)}{j\omega+0.1}e^{j\omega t}d\omega = \frac{1}{2\pi}\frac{\pi}{0.1} = 5$$

This leads to $v_o(t) = 5 - 5\text{sgn}(t) - 10e^{-t/10}u(t)$, but $\text{sgn}(t) = -1 + 2u(t)$

Therefore, $v_o(t) = 5 - 5 + 10u(t) - 10e^{-t/10}u(t)$

or $v_o(t) = \underline{\mathbf{10(1 - e^{-t/10})u(t)\ volts}}$

**Problem 17.6** *[17.29]* Determine the current i(t) in the circuit of Figure 17.5(b), given the voltage source shown in Figure 17.5(a).

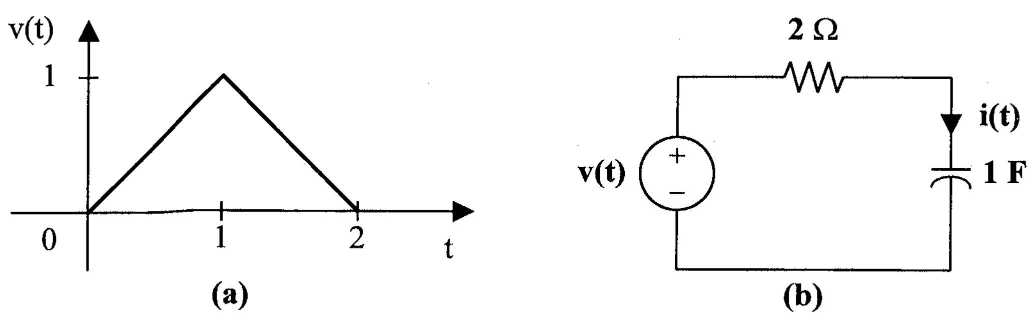



**Figure 17.5**

$$\ddot{v}(t) = \delta(t) - 2\delta(t-1) + \delta(t-2)$$

$$-\omega^2 V(\omega) = 1 - 2e^{-j\omega} + e^{j\omega 2}$$

$$V(\omega) = \frac{1 - 2e^{-j\omega} + e^{-j\omega 2}}{-\omega^2}$$

Now, $$Z(\omega) = 2 + \frac{1}{j\omega} = \frac{1 + j2\omega}{j\omega}$$

$$I = \frac{V(\omega)}{Z(\omega)} = \frac{2e^{j\omega} - e^{j\omega 2} - 1}{\omega^2} \cdot \frac{j\omega}{1 + j2\omega}$$

$$I = \frac{1}{(j\omega)(0.5 + j\omega)}(0.5 + 0.5e^{-j\omega 2} - e^{-j\omega})$$

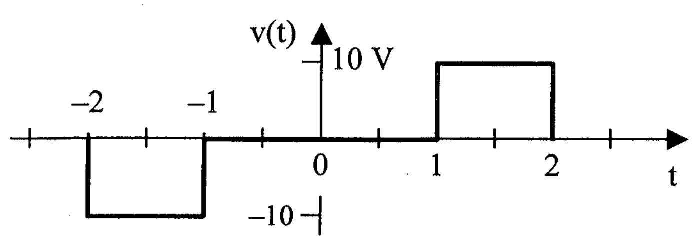
But $$\frac{1}{(s)(s+0.5)} = \frac{A}{s} + \frac{B}{s+0.5} \longrightarrow A = 2, \quad B = -2$$

$$I(\omega) = \left(\frac{2}{j\omega}\right)(0.5 + 0.5e^{j\omega 2} - e^{-j\omega}) - \left(\frac{2}{0.5 + j\omega}\right)(0.5 + 0.5e^{-j\omega 2} - e^{-j\omega})$$

$$i(t) = \underline{\frac{1}{2}\mathbf{sgn}(t) + \frac{1}{2}\mathbf{sgn}(t-2) - \mathbf{sgn}(t-1) - e^{-0.5t}\mathbf{u}(t) - e^{-0.5(t-2)}\mathbf{u}(t-2) - 2e^{-0.5(t-1)}\mathbf{u}(t-1)}$$

## PARSEVAL'S THEOREM

**Problem 17.7** Find the total energy in v(t) where v(t) is the pulse shown below.




In the time domain, $$W_{1\Omega} = \int_{-2}^{-1}(-10)^2\,dt + \int_{1}^{2}10^2\,dt = 100t\Big|_{-2}^{-1} + 100t\Big|_{1}^{2}$$

$$= -100 + 200 + 200 - 100 = \underline{\mathbf{200\ J}}$$

**Problem 17.8** *[17.43]* A voltage source $v_s(t) = e^{-t}\sin(2t)u(t)$ V is applied to a 1-Ω resistor. Calculate the energy delivered to the resistor.

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\,dt = \int_{0}^{\infty} e^{-2t}\sin^2(2t)\,dt$$

But $$\sin^2(A) = \frac{1}{2}[1-\cos(2A)]$$

$$W_{1\Omega} = \int_{0}^{\infty} e^{-2t}(0.5)[1-\cos(4t)]\,dt = \frac{1}{2}\cdot\frac{e^{-2t}}{-2}\Big|_0^{\infty} - \frac{e^{-2t}}{4+16}[-2\cos(4t)+4\sin(4t)\Big|_0^{\infty}$$

$$W_{1\Omega} = \left(\frac{1}{4}\right)+\left(\frac{1}{20}\right)(-2) = \underline{\mathbf{0.15\ J}}$$

## APPLICATIONS

**Problem 17.9** Given the AM signal,

$$f(t) = 10(1 + 4\cos(2000\pi t))\cos(\pi x10^6 t),$$

solve for the:

(a) the carrier frequency
(b) the lower sideband frequency
(c) the upper sideband frequency

$\omega_m = 2000\pi = 2\pi f$ which leads to $f = 1$ kHz

(a) $\omega_c = \pi x10^6 = 2\pi f_c$ which leads to $f_c =$ **<u>500 kHz or 0.5 MHz</u>**

(b) Lsb $= f_c - f_m = (500 - 1)$ kHz $= 499$ kHz

(c) Usb $= f_c + f_m = (500 + 1)$ kHz $= 501$ kHz

**Problem 17.10** *[17.47]* A voice signal occupying the frequency band of 0.4 to 3.5 kHz is used to amplitude modulate a 10-MHz carrier. Determine the range of frequencies for the lower and upper sidebands.

For the lower sideband, the frequencies range from

$$10{,}000{,}000 - 3{,}500 = \underline{\mathbf{9{,}996{,}500\ Hz}}$$

to

$$10{,}000{,}000 - 400 = \underline{\mathbf{9{,}999{,}600\ Hz}}$$

For the upper sideband, the frequencies range from

$$10{,}000{,}000 + 400 = \underline{\mathbf{10{,}000{,}400\ Hz}}$$

to

$$10{,}000{,}000 + 3{,}500 = \underline{\mathbf{10{,}003{,}500\ Hz}}$$