STEADY-STATE POWER ANALYSIS

LEARNING GOALS

Instantaneous Power
For the special case of steady state sinusoidal signals

Average Power
Power absorbed or supplied during one cycle

Maximum Average Power Transfer
When the circuit is in sinusoidal steady state

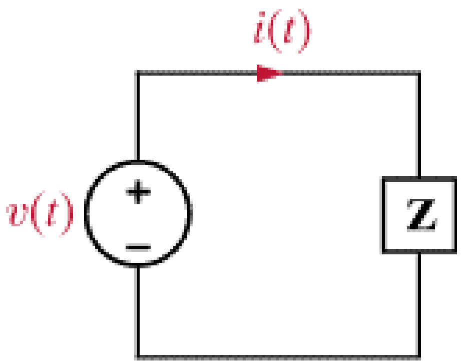
Effective or RMS Values
For the case of sinusoidal signals

Power Factor
A measure of the angle between current and voltage phasors

Power Factor Correction
How to improve power transfer to a load by “aligning” phasors

Single Phase Three-Wire Circuits
Typical distribution method for households and small loads

INSTANTANEOUS POWER



Instantaneous Power Supplied to Impedance

$$p(t) = v(t)i(t)$$

In steady State

$$v(t) = V_M \cos(\omega t + \theta_v)$$

$$i(t) = I_M \cos(\omega t + \theta_i)$$

$$p(t) = V_M I_M \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

$$\cos\phi_1 \cos\phi_2 = \frac{1}{2}[\cos(\phi_1 - \phi_2) + \cos(\phi_1 + \phi_2)]$$

$$p(t) = \frac{V_M I_M}{2}[\cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i)]$$

constant

Twice the frequency

LEARNING EXAMPLE

Assume: $v(t) = 4\cos(\omega t + 60°)$,

$$Z = 2\angle 30° \Omega$$

Find: $i(t), p(t)$

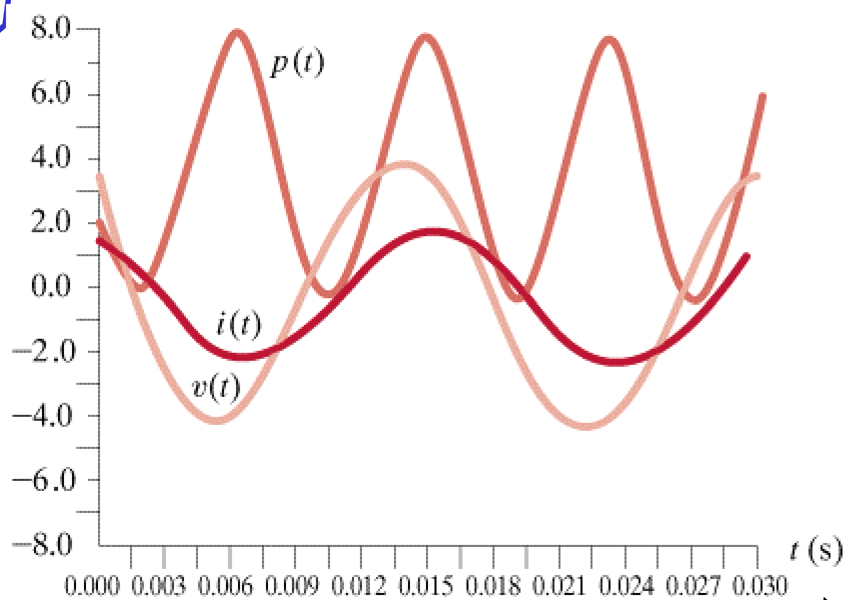
$$I = \frac{V}{Z} = \frac{4\angle 60°}{2\angle 30°} = 2\angle 30° (A)$$

$$i(t) = 2\cos(\omega t + 30°)(A)$$

$$V_M = 4, \theta_v = 60°$$

$$I_M = 2, \theta_i = 30°$$

$$p(t) = 4\cos 30° + 4\cos(2\omega t + 90°)$$



GEAUX

AVERAGE POWER

For sinusoidal (and other periodic signals) we compute averages over one period

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)dt \qquad T = \frac{2\pi}{\omega}$$

$$p(t) = \frac{V_M I_M}{2}\left[\cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i)\right]$$

$$P = \frac{V_M I_M}{2}\cos(\theta_v - \theta_i)$$

It does not matter who leads

If voltage and current are in phase

$$\theta_v = \theta_i \Rightarrow P = \frac{1}{2}V_M I_M$$

Purely resistive

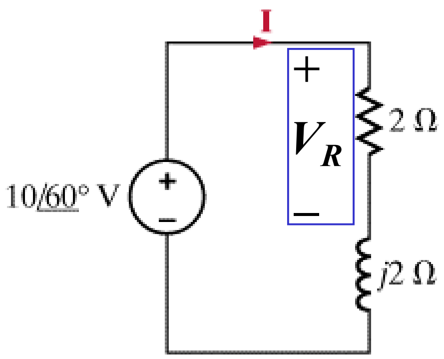
If voltage and current are in quadrature

$$\theta_v - \theta_i = \pm 90° \Rightarrow P = 0$$

Purely inductive or capacitive

LEARNING EXAMPLE

Find the average power absorbed by impedance



$$I = \frac{10\angle 60°}{2 + j2} = \frac{10\angle 60°}{2\sqrt{2}\angle 45°} = 3.53\angle 15°(A)$$

$$V_M = 10, I_M = 3.53, \theta_v = 60°, \theta_i = 15°$$

$$P = 35.3\cos(45°) = 12.5W$$

Since inductor does not absorb power one can use voltages and currents across the resistive part

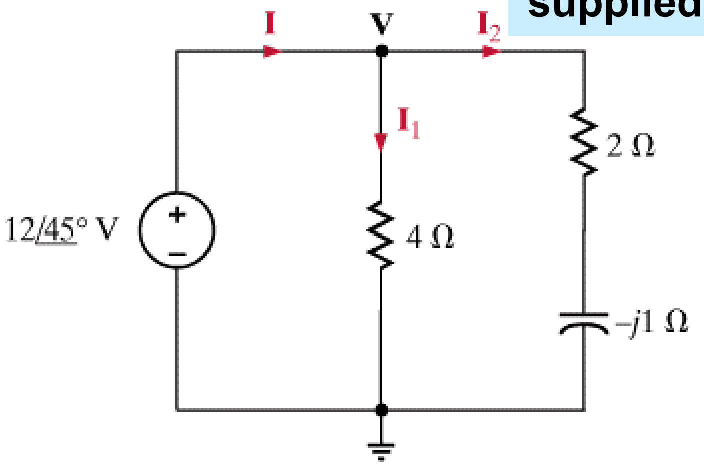
$$V_R = \frac{2}{2 + j2}10\angle 60° = 7.06\angle 15°(V)$$

$$P = \frac{1}{2}7.06 \times 3.53W$$

GEAUX

LEARNING EXAMPLE

Determine the average power absorbed by each resistor, the total average power absorbed and the average power supplied by the source



Inductors and capacitors do not absorb power in the average

$$P_{total} = 18 + 28.7W$$

$$P_{\text{supplied}} = P_{\text{absorbed}} \Rightarrow P_{\text{supplied}} = 46.7W$$

Verification

$$I = I_1 + I_2 = 3\angle 45° + 5.36\angle 71.57°$$

$$I = 8.15\angle 62.10°(A)$$

$$P = \frac{V_M I_M}{2}\cos(\theta_v - \theta_i)$$

$$P_{\text{supplied}} = \frac{1}{2} 12 \times 8.15 \times \cos(45° - 62.10°)$$

If voltage and current are in phase

$$\theta_v = \theta_i \Rightarrow P = \frac{1}{2} V_M I_M = \frac{1}{2} R I_{1M}^2 = \frac{1}{2}\frac{V_M^2}{R}$$

$$I_1 = \frac{12\angle 45°}{4} = 3\angle 45°(A)$$

$$P_{4\Omega} = \frac{1}{2} 12 \times 3 = 18W$$

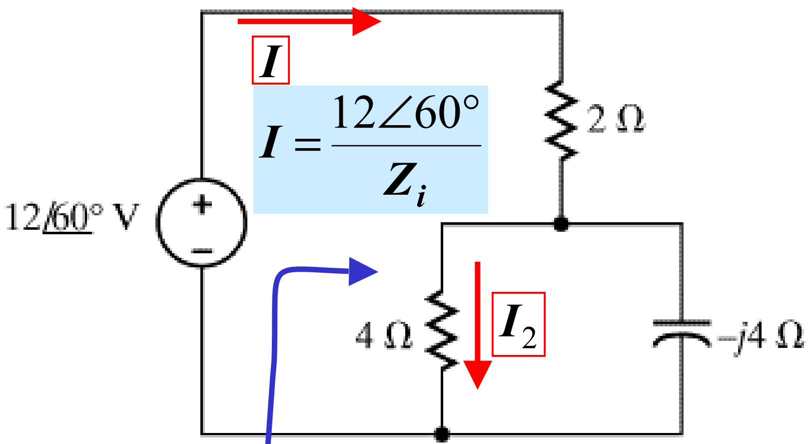
$$I_2 = \frac{12\angle 45°}{2 - j1} = \frac{12\angle 45°}{\sqrt{5}\angle -26.37°} = 5.36\angle 71.57°(A)$$

$$P_{2\Omega} = \frac{1}{2} \times 2 \times 5.36^2 (W) = 28.7W$$

GEAUX

LEARNING EXTENSION

Find average power absorbed by each resistor



$$I = \frac{12\angle 60^\circ}{Z_i}$$

$$Z_i = 2 + (4 \,||\, -j4)$$

$$Z_i = 2 + \frac{4(-j4)}{4-j4} = \frac{8-j8-j16}{4-j4} = \frac{25.3\angle -71.6^\circ}{4\sqrt{2}\angle -45^\circ}$$

$$Z_i = 4.47\angle -26.6^\circ \Omega$$

$$I = \frac{12\angle 60^\circ}{4.47\angle -26.6^\circ} = 2.68\angle 86.6^\circ (A)$$

$$P_{2\Omega} = \frac{1}{2} R I_M^2 = \frac{1}{2} \times 2 \times 2.68^2 = 7.20W$$

$$I_2 = \frac{-j4}{4-j4} I = \frac{4\angle -90^\circ}{4\sqrt{2}\angle -45^\circ} \times 2.68\angle 86.6^\circ$$

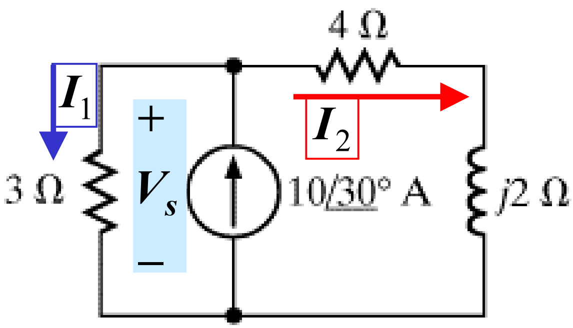
$$I_2 = 1.90\angle 41.6^\circ$$

$$P_{4\Omega} = \frac{1}{2} \times 4 \times 1.90^2 (W)$$

GEAUX

LEARNING EXTENSION

Find the AVERAGE power absorbed by each PASSIVE component and the total power supplied by the source



$$I_1 = \frac{4+j2}{3+4+j2} 10\angle 30°$$

$$I_1 = \frac{4.47\angle 26.57°}{7.28\angle 15.95°} 10\angle 30° = 6.14\angle 40.62° (A)$$

$$P_{3\Omega} = \frac{1}{2} R I_M^2 = \frac{1}{2} \times 3 \times 6.14^2 (W)$$

$$I_2 = 10\angle 30° - 6.14\angle 40.62°$$

$$I_2 = \frac{3}{3+4+j2} 10\angle 30° = \frac{30\angle 30°}{7.28\angle 15.95°}$$
$$= 4.12\angle 14.05° (A)$$

$$P_{4\Omega} = \frac{1}{2} \times 4 \times 4.12^2 (W)$$

$$P_{j2\Omega} = 0 (W)$$

Power supplied by source

Method 1. $P_{\text{supplied}} = P_{\text{absorbed}}$

$$P_{\text{supplied}} = P_{3\Omega} + P_{4\Omega} = 90.50W$$

Method 2: $P = \frac{1}{2} V_M I_M \cos(\theta_v - \theta_i)$

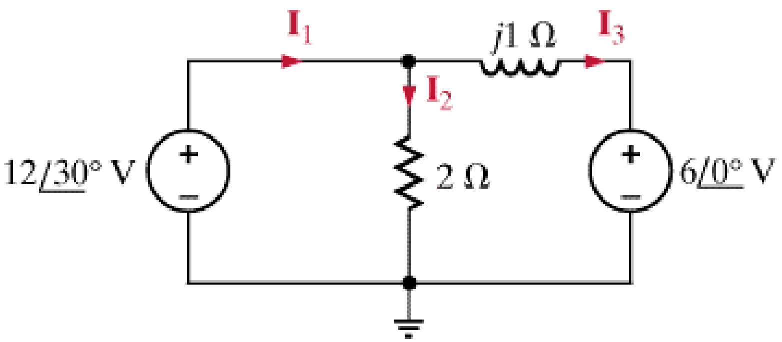
$$V_s = 3I_1 = 18.42\angle 40.62°$$

$$P = \frac{1}{2} \times 18.42 \times 10 \times \cos(40.62° - 30°)$$

GEAUX

LEARNING EXAMPLE

Determine average power absorbed or supplied by each element



$$I_3 = \frac{12\angle 30° - 6\angle 0°}{j1} = \frac{10.39 + j6 - 6}{j} = 6 - j4.39$$
$$= 7.43\angle -36.19°(A)$$

$$P_{6\angle 0°} = \frac{1}{2} \times 6 \times 7.43 \cos(0 + 36.19°) = 18W$$

Passive sign convention

$$I_2 = \frac{12\angle 30°}{2} = 6\angle 30°(A)$$

$$I_1 = I_2 + I_3 = 5.20 + j3 + 6 - j4.39 = 11.2 - j1.39(A$$
$$= 11.28\angle -7.07°$$

$$P_{2\Omega} = \frac{1}{2} R I_M^2 = \frac{1}{2} \times 2 \times 6^2 = 36(W)$$

$$P_{12\angle 30°} = -\frac{1}{2} \times 12 \times 11.28 \times \cos(30° + 7.07°)$$
$$= -54(W) \quad = -(36 + 18)$$

$$P_{j1\Omega} = 0$$

To determine power absorbed/supplied by sources we need the currents I1, I2

Average Power

$$P = \frac{1}{2} V_M I_M \cos(\theta_v - \theta_i)$$

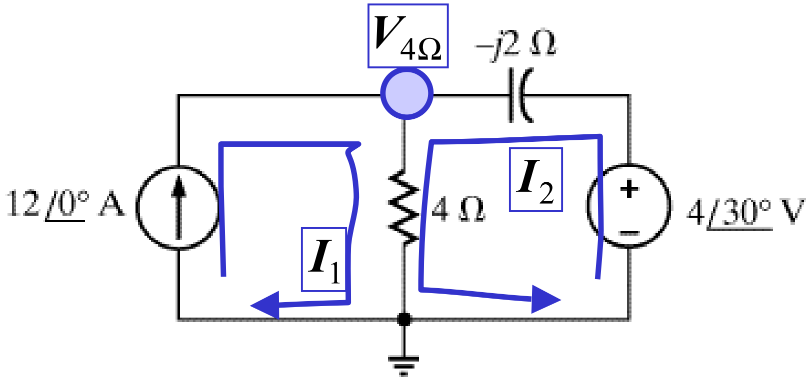
For resistors

$$P = \frac{1}{2} R I_M^2 = \frac{1}{2} \frac{V_M^2}{R}$$

GEAUX

LEARNING EXTENSION

Determine average power absorbed/supplied by each element



$$P_{12\angle 0°} = -\frac{1}{2}\times 19.92\times 12\times\cos(-54.5° - 0°) = -69.4(W)$$

$$P_{4\angle 30°} = -\frac{1}{2}\times 4\times(9.97)\cos(30° - 204°) = 19.8(W)$$

Check: Power supplied =power absorbed

Loop Equations

$$I_1 = 12\angle 0°$$

$$4\angle 30° = -j2I_2 + 4(I_2 + 12\angle 0°)$$

$$I_2 = \frac{4\angle 30° - 48\angle 0°}{4 - j2} = \frac{3.46 + j2 - 48}{4.47\angle -26.57°}$$

$$I_2 = \frac{44.58\angle 177.43°}{4.47\angle -26.57°} = 9.97\angle 204°(A)$$

$$V_{4\Omega} = 4(I_1 + I_2) = 4(12 + 9.97\angle 204°)(V)$$
$$= 4(12 - 9.108 - j4.055)(V) = 19.92\angle -54.5°(V)$$

$$P_{4\Omega} = \frac{1}{2}\frac{V_M^2}{R} = \frac{1}{2}\times\frac{19.92^2}{4} = 49.6W$$

$$P_{-2j\Omega} = 0(W)$$

Alternative Procedure

Node Equations

$$-12\angle 0° + \frac{V_{4\Omega}}{4} + \frac{V_{4\Omega} - 4\angle 30°}{-j2} = 0$$

$$I_2 = \frac{4\angle 30° - V_{4\Omega}}{-2j}$$

Average Power

$$P = \frac{1}{2}V_M I_M\cos(\theta_v - \theta_i)$$

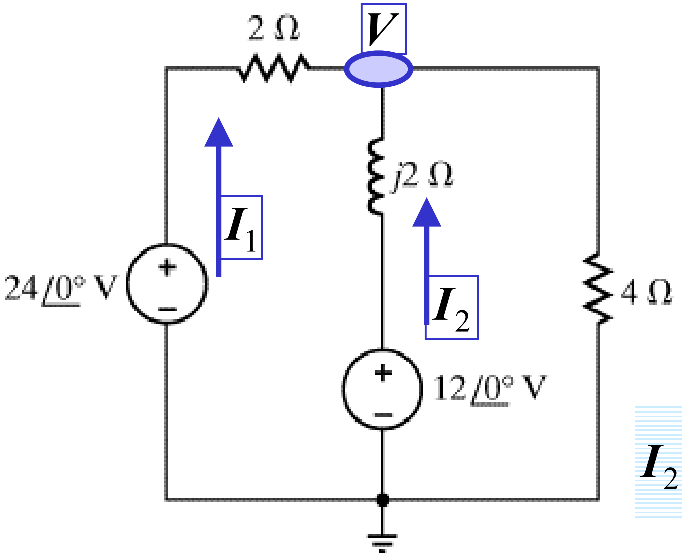
For resistors

$$P = \frac{1}{2}RI_M^2 = \frac{1}{2}\frac{V_M^2}{R}$$

GEAUX

LEARNING EXTENSION

Determine average power absorbed/supplied by each element



$$I_1 = \frac{24\angle 0^\circ - V}{2} = \frac{24 - 14.77 - j1.85}{2} = 4.62 - j0.925$$

$$I_1 = 4.71\angle -11.32^\circ (A)$$

$$I_2 = \frac{12\angle 0^\circ - V}{j2} = \frac{12 - 14.77 + j1.85}{j2} \times \frac{-j}{-j}$$

$$I_2 = \frac{-1.85 + j2.77}{2} = -0.925 + j1.385(A) = 1.67\angle 123.73^\circ (A)$$

Node Equation

$$\frac{V - 24\angle 0^\circ}{2} + \frac{V - 12\angle 0^\circ}{j2} + \frac{V}{4} = 0 \quad \times j4$$

$$2j(V - 24) + 2(V - 12) + jV = 0$$

$$(2 + 3j)V = 24 + j48$$

$$V = \frac{24 + j48}{2 + j3} \times \frac{2 - j3}{2 - j3} = \frac{192 + j24}{13}$$

$$= 14.88\angle 7.125^\circ (V)$$

$$= 14.77 + j1.85(V)$$

$$P_{2\Omega} = \frac{1}{2} \times 2 \times 4.71^2 = 22.18(W)$$

$$P_{4\Omega} = \frac{1}{2} \times \frac{14.88^2}{4} = 27.67(W)$$

For resistors

$$P = \frac{1}{2} R I_M^2 = \frac{1}{2} \frac{V_M^2}{R}$$

$$P_{12\angle 0^\circ} = -\frac{1}{2} \times 12 \times 1.67 \cos(0^\circ - 123.73^\circ) = 5.565(W)$$

$$P_{24\angle 0^\circ} = -\frac{1}{2} \times 24 \times 4.71 \times \cos(0^\circ + 11.32^\circ) = -55.42(W)$$

Average Power

$$P = \frac{1}{2} V_M I_M \cos(\theta_v - \theta_i)$$

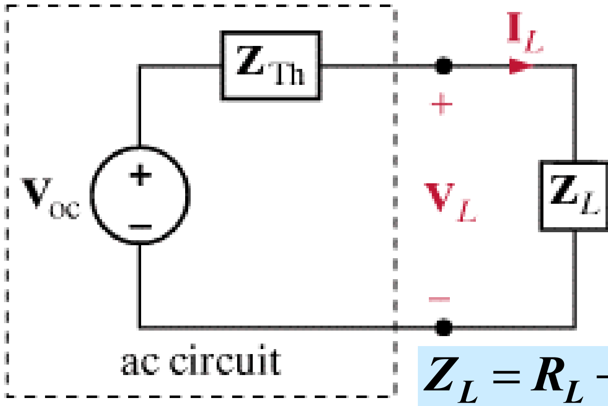
Check :

$$P_{\text{absorbed}} = 22.18 + 27.67 + 5.565(W)$$

$$P_{\text{supplied}} = 55.42(W)$$

MAXIMUM AVERAGE POWER TRANSFER

$$Z_{TH} = R_{TH} + jX_{TH}$$



$$Z_L = R_L + jX_L$$

$$P_L = \frac{1}{2} V_{LM} I_{LM} \cos(\theta_{V_L} - \theta_{I_L})$$

$$= \frac{1}{2} |V_L| |I_L| \cos(\theta_{V_L} - \theta_{I_L})$$

$$V_L = \frac{Z_L}{Z_L + Z_{TH}} V_{OC} \quad \Rightarrow |V_L| = \left| \frac{Z_L}{Z_L + Z_{TH}} \right| |V_{OC}|$$

$$I_L = \frac{V_L}{Z_L} \quad \Rightarrow \angle I_L = \angle V_L - \angle Z_L \quad \Rightarrow |I_L| = \frac{|V_L|}{|Z_L|}$$

$$\Rightarrow \theta_{V_L} - \theta_{I_L} = \angle Z_L$$

$$Z_L = R_L + jX_L \quad \Rightarrow \tan(\angle Z_L) = \frac{X_L}{R_L}$$

$$\cos\theta = \frac{1}{\sqrt{1 + \tan^2\theta}}$$

$$\therefore \cos(\theta_{V_L} - \theta_{I_L}) = \frac{R_L}{\sqrt{R_L^2 + X_L^2}}$$

$$P_L = \frac{1}{2} \frac{|Z_L| |V_{OC}|^2}{|Z_L + Z_{TH}|^2} \frac{R_L}{\sqrt{R_L^2 + X_L^2}}$$

$$Z_L + Z_{TH} = (R_L + R_{TH}) + j(X_L + X_{TH})$$

$$|Z_L + Z_{TH}|^2 = (R_L + R_{TH})^2 + (X_L + X_{TH})^2$$

$$P_L = \frac{1}{2} \frac{|V_{OC}|^2 R_L}{(R_L + R_{TH})^2 + (X_L + X_{TH})^2}$$

$$\left. \begin{array}{l} \dfrac{\partial P_L}{\partial X_L} = 0 \\ \dfrac{\partial P_L}{\partial R_L} = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} X_L = -X_{TH} \\ R_L = R_{TH} \end{array} \right.$$

$$\therefore Z_L^{opt} = Z_{TH}^*$$

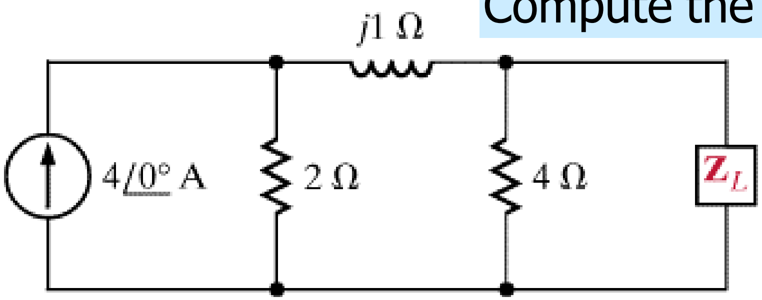
$$P_L^{\max} = \frac{1}{2} \left(\frac{|V_{OC}|^2}{4R_{TH}} \right)$$

GEAUX

LEARNING EXAMPLE

Find Z_L for maximum average power transfer.

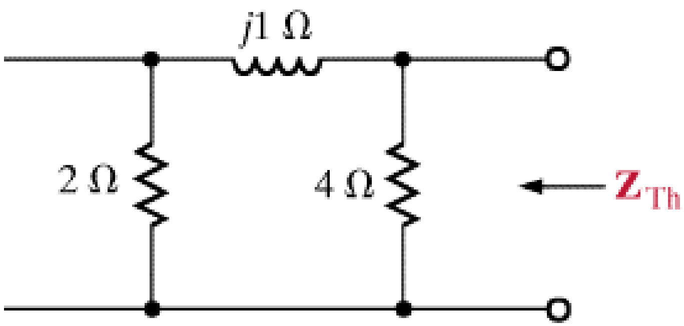
Compute the maximum average power supplied to the load



$$\therefore Z_L^{opt} = Z_{TH}^{*}$$

$$P_L^{\max} = \frac{1}{2}\left(\frac{|V_{OC}|^2}{4R_{TH}}\right)$$

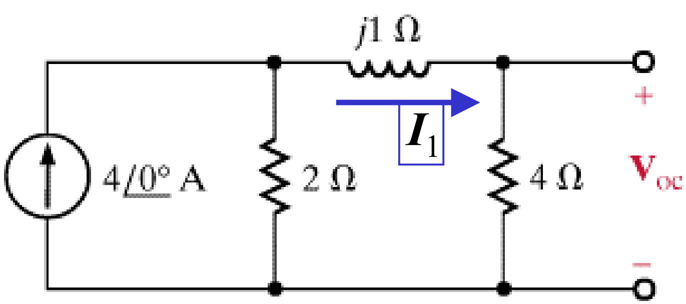
Remove the load and determine the Thevenin equivalent of remaining circuit



$$Z_{TH} = 4 \,||\, (2 + j1) = \frac{8+j4}{6+j1} = \frac{(8+j4)(6-j1)}{37} = \frac{52+j16}{37}\Omega$$

$$= \frac{8+j4}{6+j1} = \frac{8.94\angle 26.57^\circ}{6.08\angle 9.64^\circ} = 1.47\angle 16.93^\circ\Omega$$

$$Z_L^{*} = 1.47\angle -16.93^\circ = 1.41 - j0.43\Omega$$



$$V_{OC} = 4 \times \frac{2}{2+4+j1} 4\angle 0^\circ = \frac{32\angle 0^\circ}{6.08\angle 9.64^\circ} = 5.26\angle -9.64^\circ$$

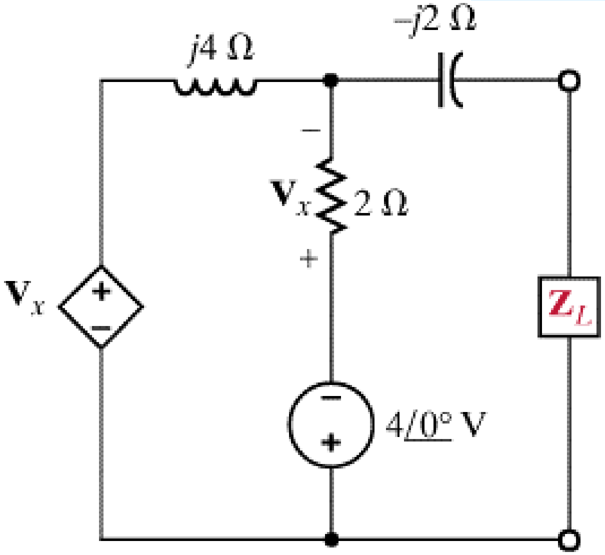
$$P_L^{\max} = \frac{1}{2} \times \frac{5.26^2}{4 \times 1.41} = 2.45(W)$$

We are asked for the value of the power. We need the Thevenin voltage

LEARNING EXAMPLE

Find Z_L for maximum average power transfer.

Compute the maximum average power supplied to the load

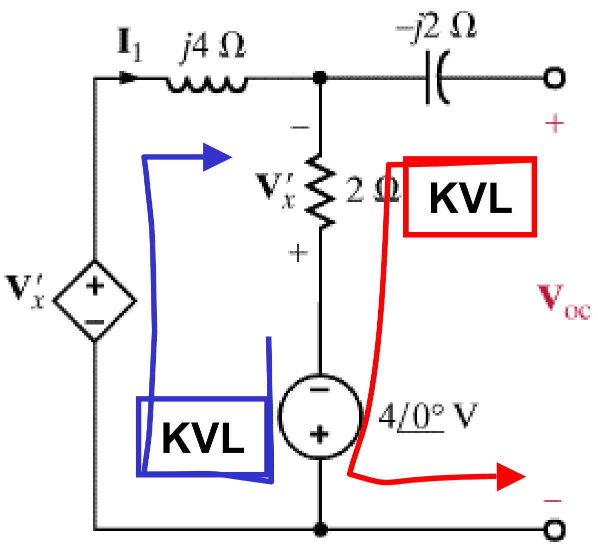


$$\therefore Z_L^{opt} = Z_{TH}^*$$

$$P_L^{\max} = \frac{1}{2}\left(\frac{|V_{OC}|^2}{4R_{TH}}\right)$$

Circuit with dependent sources!

$$Z_{TH} = \frac{V_{OC}}{I_{SC}}$$



$$4\angle 0° = -V_x' + (2 + j4)I_1$$

$$V_X' = -2I_1$$

$$4\angle 0° = (4 + j4)I_1 = (4\sqrt{2}\angle 45°)I_1$$

$$I_1 = \frac{4\angle 0°}{4\sqrt{2}\angle 45°} = 0.707\angle -45°(A)$$

$$V_{OC} = 2I_1 - 4\angle 0° = 1 - j1 - 4 = -3 - j1 = \sqrt{10}\angle -161.5°$$

Next: the short circuit current ...

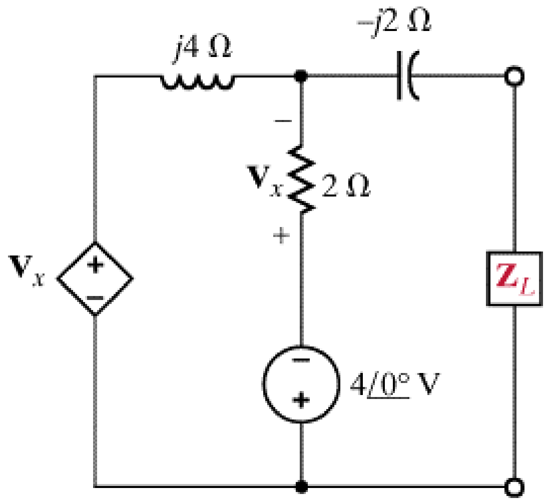
GEAUX

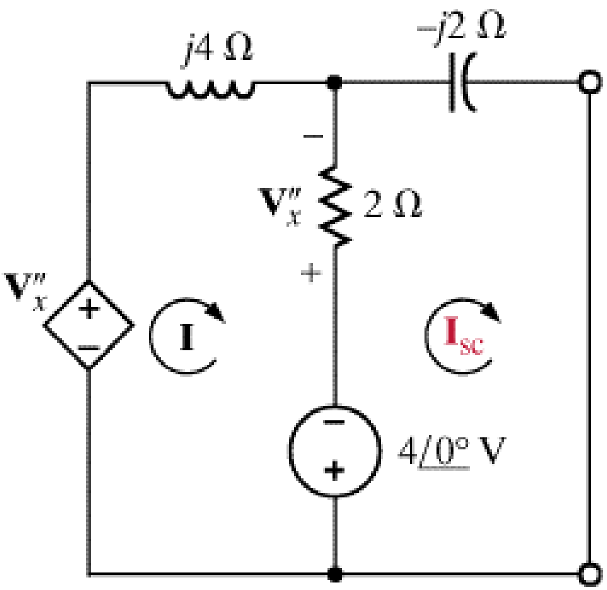
LEARNING EXAMPLE (continued)...

$$\therefore Z_L^{opt} = Z_{TH}^*$$

$$P_L^{\max} = \frac{1}{2}\left(\frac{|V_{OC}|^2}{4R_{TH}}\right)$$

Original circuit





LOOP EQUATIONS FOR SHORT CIRCUIT CURRENT

$$-V_x'' + j4I + 2(I - I_{SC}) - 4\angle° = 0$$

$$4\angle 0° + 2(I_{SC} - I) - j2I_{SC} = 0$$

CONTROLLING VARIABLE

$$V_x'' = 2(I_{SC} - I)$$

Substitute and rearrange

$$(4 + j4)I - 4I_{SC} = 4$$

$$-2I + (2 - j2)I_{SC} = -4 \Rightarrow I = (1 - j1)I_{SC} + 2$$

$$4(1 + j)[(1 - j)I_{SC} + 2] - 4I_{SC} = 4$$

$$I_{SC} = -1 - j2(A) = \sqrt{5}\angle -116.57°$$

$$V_{OC} = 2I_1 - 4\angle 0° = 1 - j1 - 4 = -3 - j1 = \sqrt{10}\angle -161.57°$$

$$Z_{TH} = \sqrt{2}\angle -45° = 1 - j1\Omega \Rightarrow Z_L^{opt} = 1 + j1\Omega$$

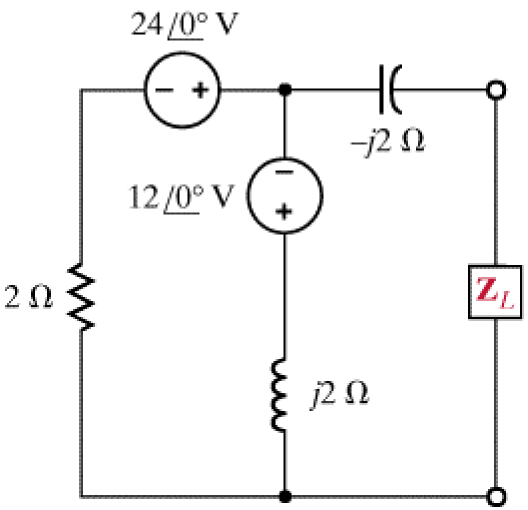
$$P_L^{\max} = \frac{1}{2} \times \frac{(\sqrt{10})^2}{4} = 1.25(W)$$

GEAUX

LEARNING EXTENSION

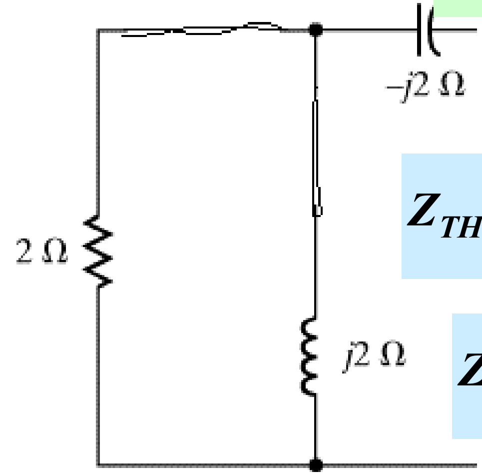
Find Z_L for maximum average power transfer.

Compute the maximum average power supplied to the load



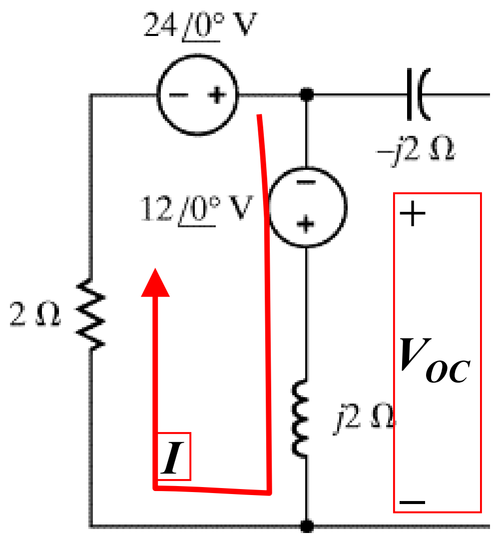
$$\therefore Z_L^{opt} = Z_{TH}^*$$

$$P_L^{\max} = \frac{1}{2}\left(\frac{|V_{OC}|^2}{4R_{TH}}\right)$$



$$Z_{TH} = -j2 + (2 \parallel j2) = -j2 + \frac{4j}{2+j2}\,\Omega$$

$$Z_{TH} = \frac{4}{2+j2} = \frac{8-j8}{8} = 1 - j(\Omega)$$



$$V_{OC} = -12\angle 0^\circ + j2I$$
$$= -12 + j2 \times 9(1-j)$$
$$= 6 + j18$$

$$Z_L^{opt} = 1 + j(\Omega)$$

$$V_{OC} = 18.974\angle 71.57^\circ (V)$$

$$|V_{OC}|^2 = 6^2 + 18^2 = 360$$

$$P_L^{\max} = \frac{1}{2} \times \frac{360}{4} = 45(W)$$

$$36\angle 0^\circ = (2+j2)I$$

$$I = \frac{36(2-j2)}{8} = 9(1-j) = 12.73\angle -45^\circ$$

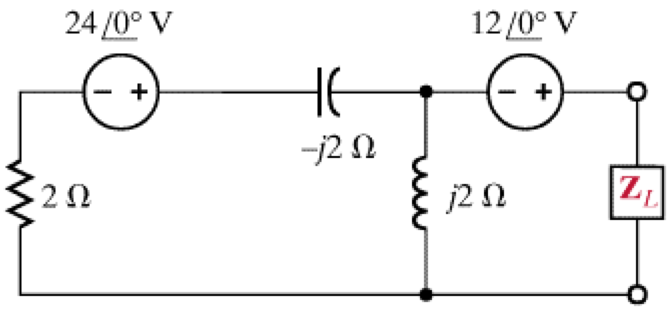
LEARNING EXTENSION

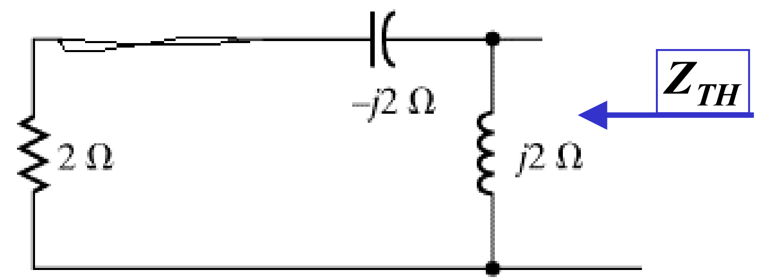
Find Z_L for maximum average power transfer.

Compute the maximum average power supplied to the load

$$\therefore Z_L^{opt} = Z_{TH}^*$$

$$P_L^{\max} = \frac{1}{2}\left(\frac{|V_{OC}|^2}{4R_{TH}}\right)$$



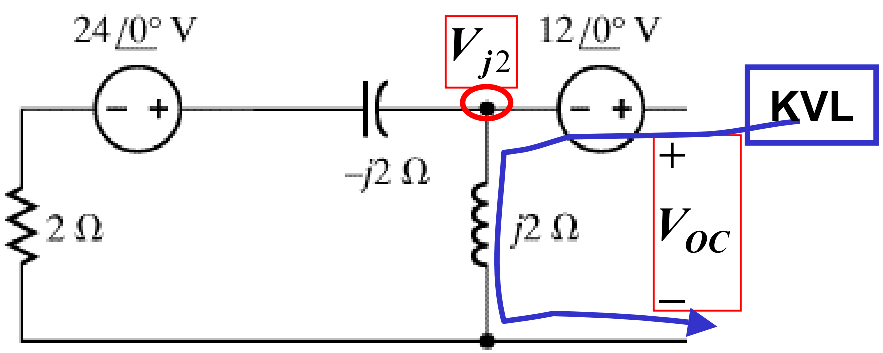


$$Z_{TH} = j2 \,\|\, (2 - j2) = \frac{j2(2 - j2)}{2 + j2 - j2}$$

$$Z_{TH} = 2 + j2(\Omega)$$

$$Z_L^{opt} = 2 - j2(\Omega)$$

$$P_L^{\max} = \frac{1}{2} \times \frac{720}{4 \times 2} = 45(W)$$



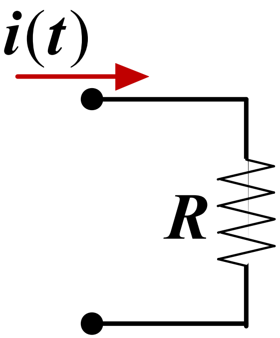
$$V_{j2} = \frac{j2}{j2 - j2 + 2} 24\angle 0^\circ = 24\angle 90^\circ$$

$$V_{OC} = 12\angle 0^\circ + 24\angle 90^\circ = 12 + j24(V)$$

$$|V_{OC}|^2 = 12^2 + 24^2 = 720$$

GEAUX

EFFECTIVE OR RMS VALUES



Instantaneous power

$$p(t) = i^2(t)R$$

The effective value is the equivalent DC value that supplies the same average power

If current is periodic with period T

$$P_{av} = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)dt = R\left(\frac{1}{T}\int_{t_0}^{t_0+T} i^2(t)dt\right)$$

If current is DC ($i(t) = I_{dc}$) then

$$P_{dc} = RI_{dc}^2$$

$$I_{eff} : P_{av} = P_{dc}$$

$$I_{eff}^2 = \frac{1}{T}\int_{t_0}^{t_0+T} i^2(t)dt$$

$$I_{eff} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} i^2(t)dt}$$

Definition is valid for ANY periodic signal with period T

If the current is sinusoidal the average power is known to be

$$P_{av} = \frac{1}{2} I_M^2 R$$

$$\therefore I_{eff}^2 = \frac{1}{2} I_M^2$$

For a sinusoidal signal

$$x(t) = X_M \cos(\omega t + \theta)$$

the effective value is

$$X_{eff} = \frac{X_M}{\sqrt{2}}$$

For sinusoidal case $P_{av} = \frac{1}{2} V_M I_M \cos(\theta_v - \theta_i)$

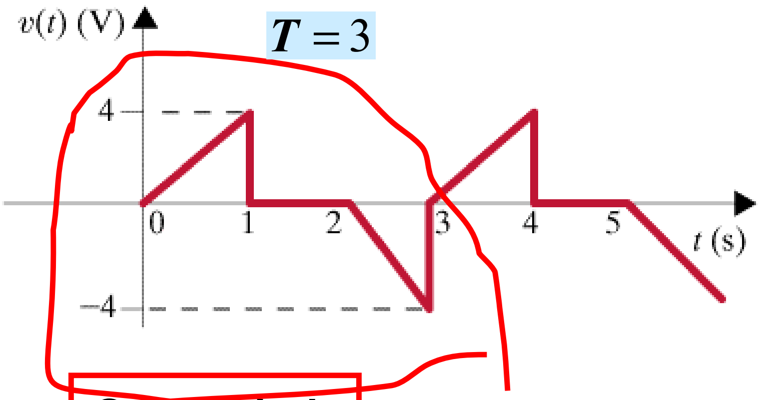
$$P_{av} = V_{eff} I_{eff} \cos(\theta_v - \theta_i)$$

effective $\approx$ rms (root mean square)

GEAUX

LEARNING EXAMPLE

Compute the rms value of the voltage waveform



$$X_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)dt}$$

$$v(t) = \begin{cases} 4t & 0 < t \le 1 \\ 0 & 1 < t \le 2 \\ -4(t-2) & 2 < t \le 3 \end{cases}$$

The two integrals have the same value

$$\int_0^T v^2(t)dt = \int_0^1 (4t)^2 dt + \int_2^3 (4(t-2))^2 dt$$

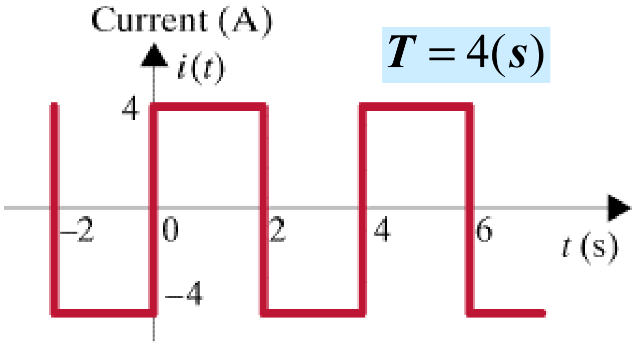
$$\int_0^3 v^2(t)dt = 2\times\left[\frac{16}{3}t^3\right]_0^1 = \frac{32}{3}$$

$$V_{rms} = \sqrt{\frac{1}{3}\times\frac{32}{3}} = 1.89(V)$$

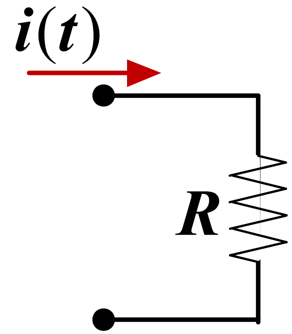
GEAUX

LEARNING EXAMPLE

Compute the rms value of the voltage waveform and use it to determine the average power supplied to the resistor



$T = 4(s)$



$R = 2\Omega$

$$X_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)dt}$$

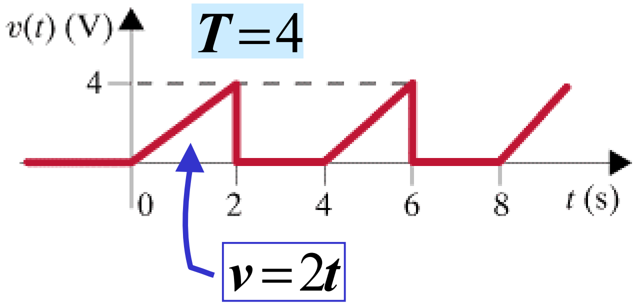
$i^2(t) = 16;\ 0 \le t < 4$

$I_{rms} = 4(A)$

$$P_{av} = RI_{rms}^2 = 32(W)$$

GEAUX

LEARNING EXTENSION Compute rms value of the voltage waveform



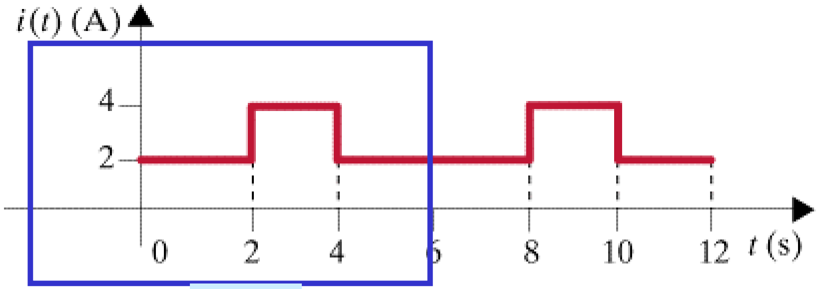
$$X_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)\,dt}$$

$$V_{rms} = \sqrt{\frac{1}{4}\int_0^2 (2t)^2 dt} = \sqrt{\left[\frac{1}{3}t^3\right]_0^2} = \sqrt{\frac{8}{3}}(V) = 1.633V$$

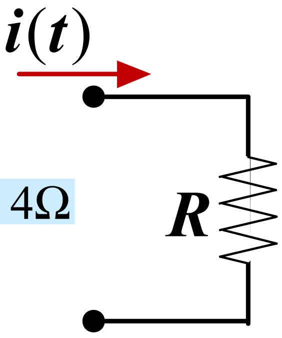
GEAUX

LEARNING EXTENSION

Compute the rms value for the current waveforms and use them to determine average power supplied to the resistor



$T = 6$



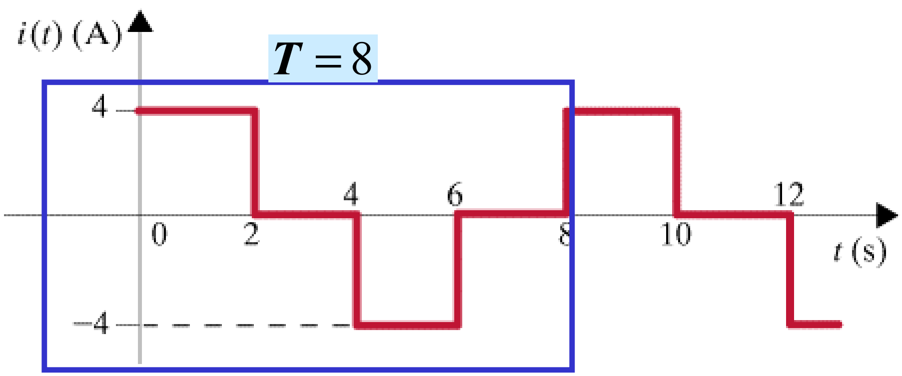
$R = 4\Omega$

$$X_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)dt}$$

$$P_{av} = I_{rms}^2 R$$

$$I_{rms}^2 = \frac{1}{6}\left[\int_0^2 4dt + \int_2^4 16dt + \int_4^6 4dt\right] = \frac{8+32+8}{6} = 8$$

$$P = 8 \times 4 = 32(W)$$



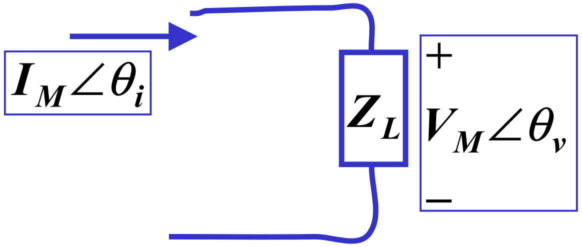
$T = 8$

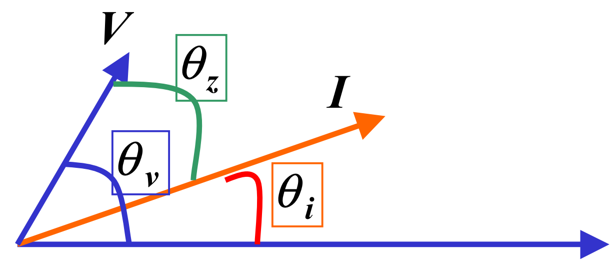
$$I_{rms}^2 = \frac{1}{8}\left[\int_0^2 16dt + \int_4^6 16dt\right] = 8$$

$$P = 32(W)$$

GEAUX

THE POWER FACTOR





$$V = ZI \Rightarrow \angle V = \angle Z + \angle I$$

$$\theta_v = \theta_z + \theta_i$$

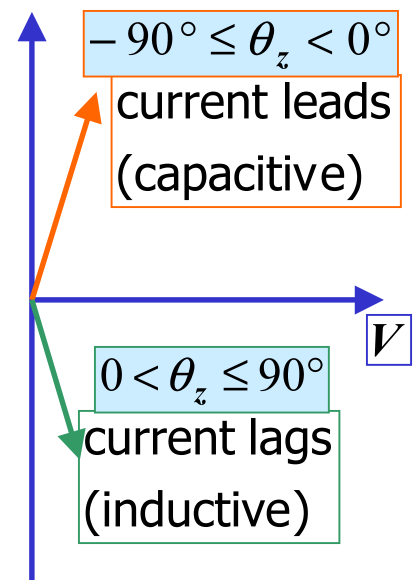
$$P = \frac{1}{2} V_M I_M \cos(\theta_v - \theta_i) = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$$

$$P_{\text{apparent}} = V_{rms} I_{rms}$$

$$pf = \frac{P}{P_{\text{apparent}}} = \cos(\theta_v - \theta_i) = \cos\theta_z$$

$$P = V_{rms} \times I_{rms} \times pf$$

pf	θ_z	
0	$-90°$	pure capacitive
$0 < pf < 1$	$-90° < \theta_z < 0°$	leading or capacitive
1	$0°$	resistive
$0 < pf < 1$	$0° < \theta_z < 90°$	lagging or inductive
0	$90°$	pure inductive

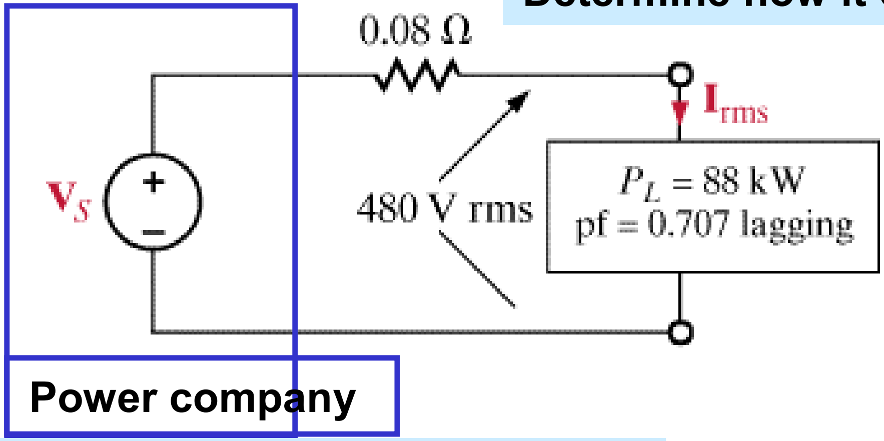


GEAUX

LEARNING EXAMPLE

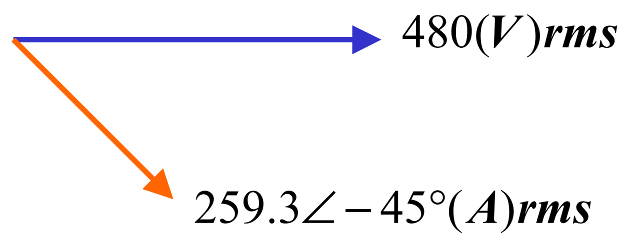
Find the power supplied by the power company.
Determine how it changes if the power factor is changed to 0.9

$$P = V_{rms} \times I_{rms} \times pf$$



$$\Rightarrow \cos\theta_z = 0.707 \Rightarrow \theta_z = -45°$$

Current lags the voltage



$$I_{rms} = \frac{88\times10^3 (W)}{480\times0.707} = 259.3(A)rms$$

$$P_{losses} = I_{rms}^2 R = 259.3^3 \times 0.08 = 5.378kW$$

$$P_S = P_{losses} + 88{,}000(W) = 93.378(kW)$$

$$V_{S_{rms}} = 0.08 I_{rms} + V_L$$
$$= 0.08\times259.3\angle-45° + 480$$

$$V_{S_{rms}} = 0.08\times(183.4 - j183.4) + 480$$
$$= 494.7 - j14.7 = 495\angle-1.7°(V)$$

If pf=0.9

$$I_{rms} = \frac{88{,}000}{480\times0.9} = 203.7(A)rms$$

$$P_{losses} = I_{rms}^2 R = 3.32kW$$

If pf=0.9

$$I_{rms} = 203.7\angle-25.8°$$

$$V_S = 14.47 - j7.09 + 480 = 494\angle-0.82°$$

Losses can be reduced by 2kW!

Examine also the generated voltage

GEAUX

LEARNING EXTENSION

$P_L = 100kW, V_L = 480(V)rms, pf = 0.707$

$R_{line} = 0.1\Omega$

$P = V_{rms} \times I_{rms} \times pf$

Determine the power savings if the power factor can be increased to 0.94

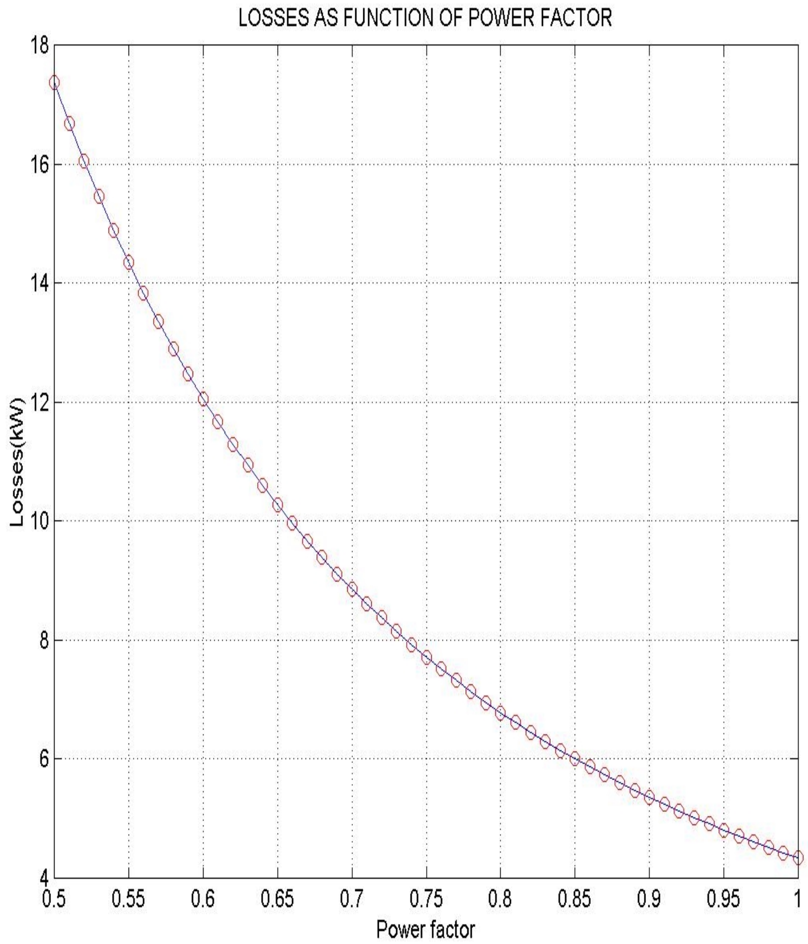
$$I_{rms} = \frac{P}{V_{rms} \times pf}$$

$$P_{losses} = I_{rms}^2 R_{line} = \frac{P^2 R_{line}}{V_{rms}^2} \times \frac{1}{pf^2}$$

$$P_{losses}(pf = 0.707) = \frac{10^{10} \times 0.1}{480^2} \times \frac{1}{0.707^2}(W)$$
$$= 2 \times 4.34kW$$

$$P_{losses}(pf = 0.94) = \frac{10^{10} \times 0.1}{480^2} \times \frac{1}{0.94^2}(W)$$
$$= 1.13 \times 4.34kW$$

$$P_{saved} = 0.87 \times 4.34kW = 3.77kW$$

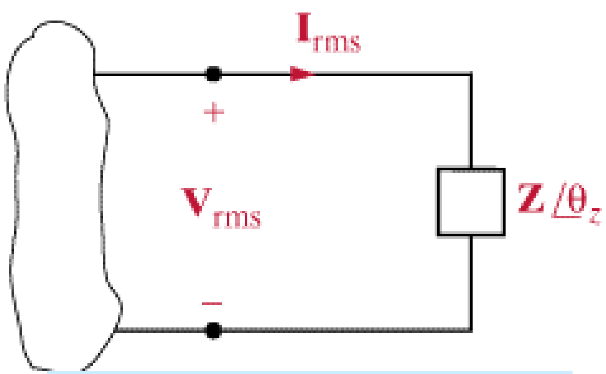


GEAUX

COMPLEX POWER

Definition of Complex Power

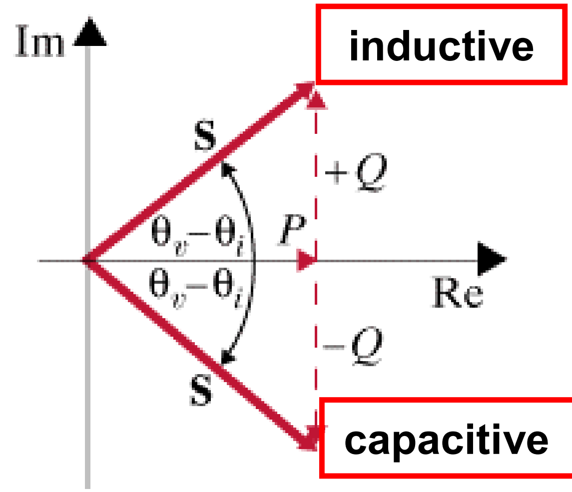
$$S = V_{rms} I_{rms}^*$$



The units of apparent and reactive power are Volt-Ampere

$$S = V_{rms}\angle\theta_v° \times [I_{rms}\angle\theta_i°]^*$$

$$S = V_{rms} I_{rms} \angle \theta_v - \theta_i$$



$$S = V_{rms} I_{rms} \cos(\theta_v - \theta_i) + j V_{rms} I_{rms} \sin(\theta_v - \theta_i)$$

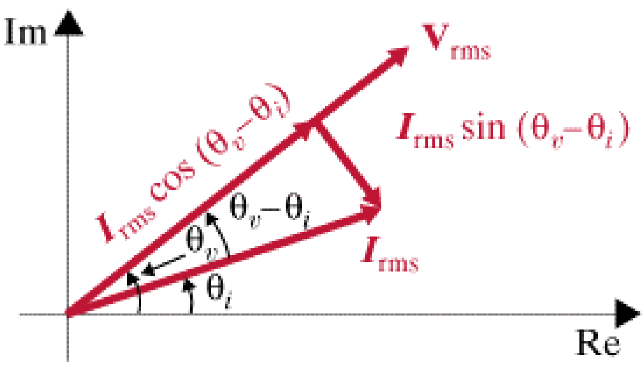
P — **Active Power**

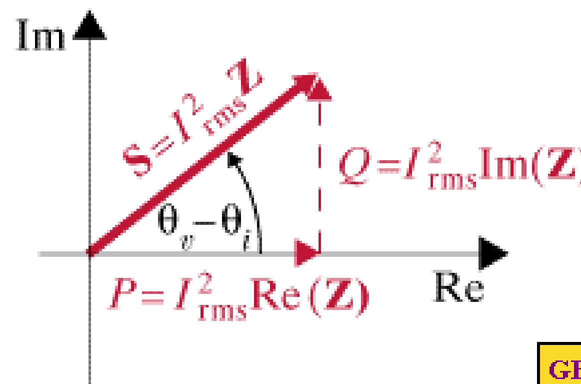
Q — **Reactive Power**

Another useful form

$$V_{rms} = Z I_{rms} \Rightarrow S = (Z I_{rms}) I_{rms}^* = Z |I_{rms}|^2$$

$$Z = R + jX \Rightarrow \begin{cases} P = R|I_{rms}|^2 \\ Q = X|I_{rms}|^2 \end{cases}$$





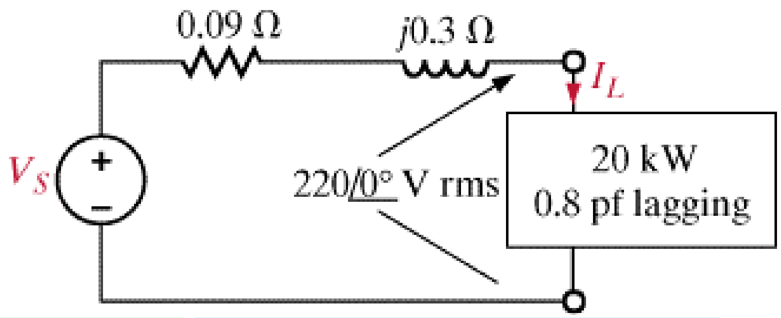
GEAUX

LEARNING EXAMPLE

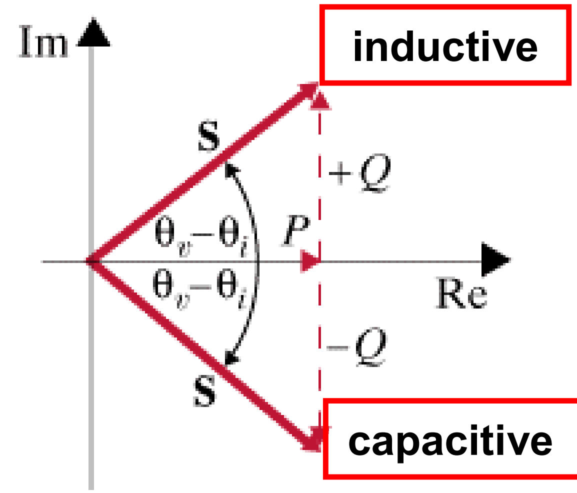
Given:

$P_L = 20kW$, $pf = 0.8\,lagging$, $V_L = 220\angle 0°\,rms$, $Z_L = 0.09 + j0.3\Omega$, $f = 60Hz$

Determine the voltage and power factor at the input to the line



inductive



$P = \mathrm{Re}\{S\} = |S|\cos(\theta_v - \theta_i) = |S| \times pf$

$\therefore |S_L| = \frac{P}{pf} = 25kVA$

$Q^2 = S_L^{\,2} - P^2 \Rightarrow Q = 15kVA$

$S_L = 20 + j15(kVA) = 25\angle 36.87°$

$S_L = V_L I_L^*$

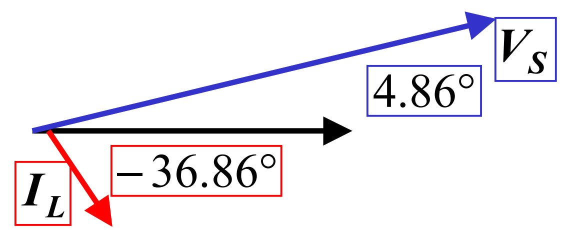
$\Rightarrow I_L = \left[\frac{S_L}{V_L}\right]^* = \left[\frac{25{,}000\angle 36.87°}{220\angle 0°}\right]^* = 113.64\angle -36.86°(A)$

$I_L = \left[\frac{20{,}000 + j15{,}000}{220}\right]^* = 90.91 - j68.18(A)$

$V_S = (0.09 + j0.3)I_L + 220\angle 0°$

$V_S = (0.09 + j0.3)(90.91 - j68.18) + 220(V)$

$V_S = 248.63 + j21.14 = 249.53\angle 4.86°$



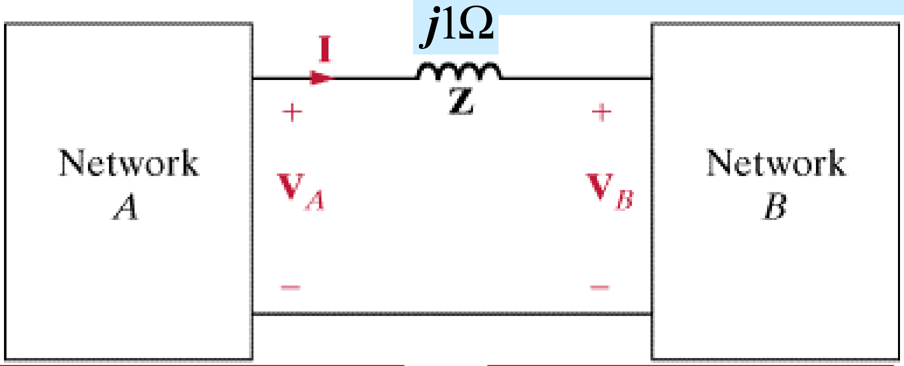
$pf_{\text{source}} = \cos(41.72°) = 0.746$

lagging

GEAUX

LEARNING EXAMPLE

Compute the average power flow between networks
Determine which is the source



$V_A = 120\angle 30°(V)rms$ $V_B = 120\angle 0°(V)rms$

$$I = \frac{V_A - V_B}{Z} = \frac{120\angle 30° - 120\angle 0°}{j1}$$

$$I = \frac{(103.92 + j60) - 120}{j} = 60 + j16.08(A)rms$$

$$I = 62.12\angle 15°(A)rms$$

$$S_A = V_A(-I)^* = 120\angle 30° \times 62.12\angle -195° = 7{,}454\angle -165° VA_{rms}$$

Passive sign convention.
Power received by A

$$P_A = 7{,}454\cos(165°) = -7{,}200(W)$$

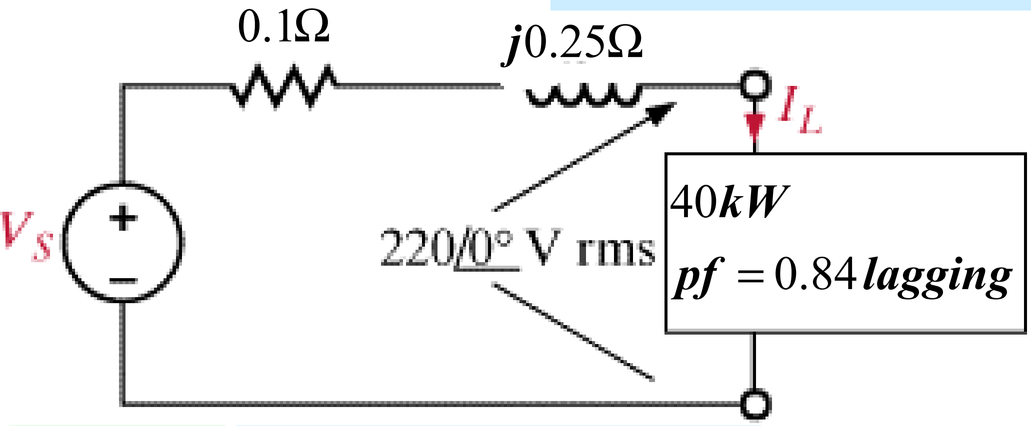
$$S_B = V_B(I)^* = 120\angle 0° \times 62.12\angle -15° = 7{,}454\angle -15° VA_{rms}$$

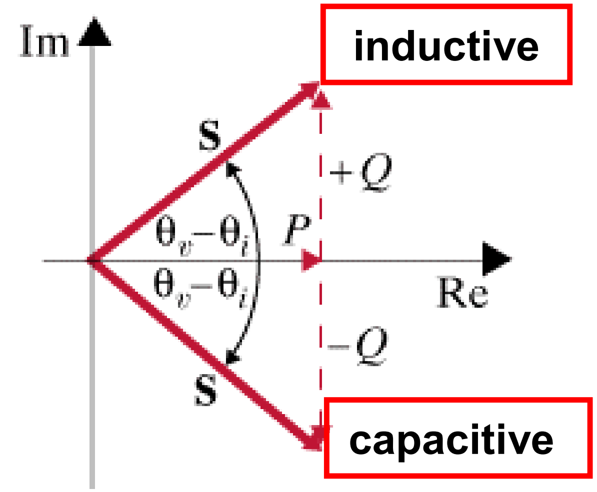
$$P_B = 7{,}454\cos(-15°) = 7{,}200(W)$$

A supplies 7.2kW average power to B

LEARNING EXTENSION

Determine real and reactive power losses and real and reactive power supplied



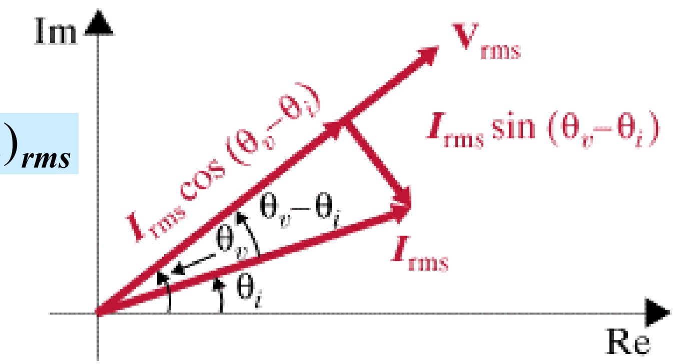


$P = \mathrm{Re}\{S\} = |S|\cos(\theta_v - \theta_i) = |S| \times pf$

$|S_L| = \frac{P}{pf} = \frac{40}{.84} = 47.62kVA$ $\quad |Q_L| = \sqrt{|S_L|^2 - P^2} = 25{,}839(VA)$

$S = VI^* \Rightarrow |I_L| = \frac{|S_L|}{|V_L|} = 216.45(A)_{rms}$

$pf = \cos(\theta_v - \theta_i) \Rightarrow \theta_v - \theta_i = 32.86°$ $\quad I_L = 216.45\angle -32.86°(A)_{rms}$



$S_{\text{losses}} = (Z_{line} I_L) I_L^* = Z_{line} |I_L|^2$

$S_{\text{losses}} = (0.1 + j0.25)(216.45)^2 = 4{,}685 + j11{,}713VA$

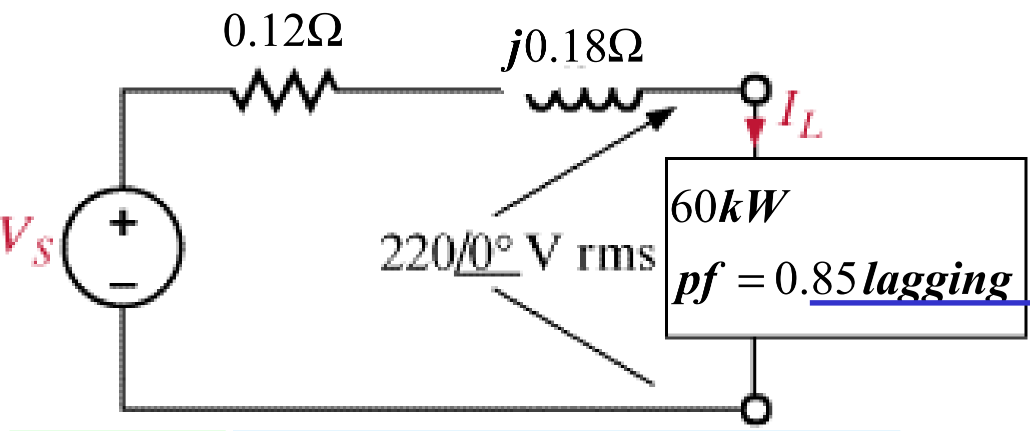
Balance of power

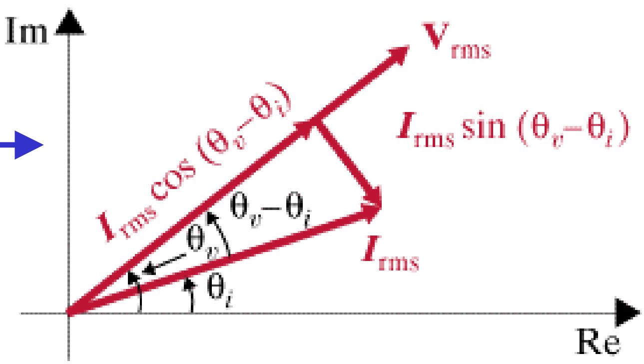
$$S_{\text{supplied}} = S_{\text{losses}} + S_{\text{load}}$$
$$= 4.685 + j11.713 + 40 + j25.839 = 44.685 + j37.552kVA$$

GEAUX

LEARNING EXTENSION

Determine line voltage and power factor at the supply end





$$P = \text{Re}\{S\} = |S|\cos(\theta_v - \theta_i) = |V_L| \times |I_L| \times pf$$

$$S_L = V_L I_L^*$$

$$|I_L| = \frac{P}{|V_L| \times pf} = 320.86(A)_{rms}$$

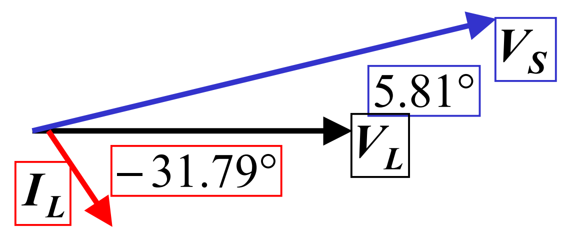
$$\theta_v - \theta_i = \cos^{-1}(pf) \Rightarrow \theta_v - \theta_i = 31.79°$$

$$I_L = 320.86\angle -31.79°(A)_{rms} = 272.72 - j169.03(A)_{rms}$$

$$V_S = Z_{\text{line}} I_L + V_L = (0.12 + j0.18)(272.72 - j169.03) + 220$$

$$V_S = 283.15 + j28.81(V)_{rms} = 284.61\angle 5.81°(V)_{rms}$$

The phasor diagram helps in visualizing the relationship between voltage and current

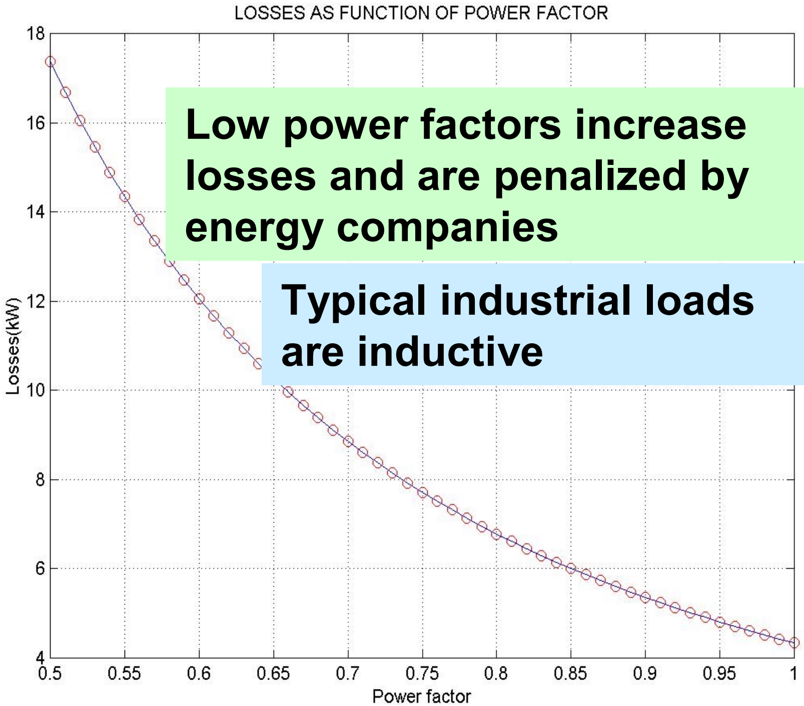


$$pf_{\text{source}} = \cos(37.6°) = 0.792$$

lagging

GEAUX

POWER FACTOR CORRECTION



Low power factors increase losses and are penalized by energy companies

Typical industrial loads are inductive

$$V_L = \frac{1}{j\omega C} I_{\text{capacitor}}$$

Without capacitor :

$$S_{\text{old}} = P_{\text{old}} + jQ_{\text{old}} = |S_{\text{old}}| \angle \theta_{\text{old}}$$

$$pf_{\text{old}} = \cos(\theta_{\text{old}})$$

With capacitor

$$S_{\text{new}} = S_{\text{old}} + S_{\text{capacitor}}$$

$$= P_{\text{old}} + jQ_{\text{old}} - jQ_{\text{capacitor}}$$

$$= |S_{\text{new}}| \angle \theta_{\text{new}}$$

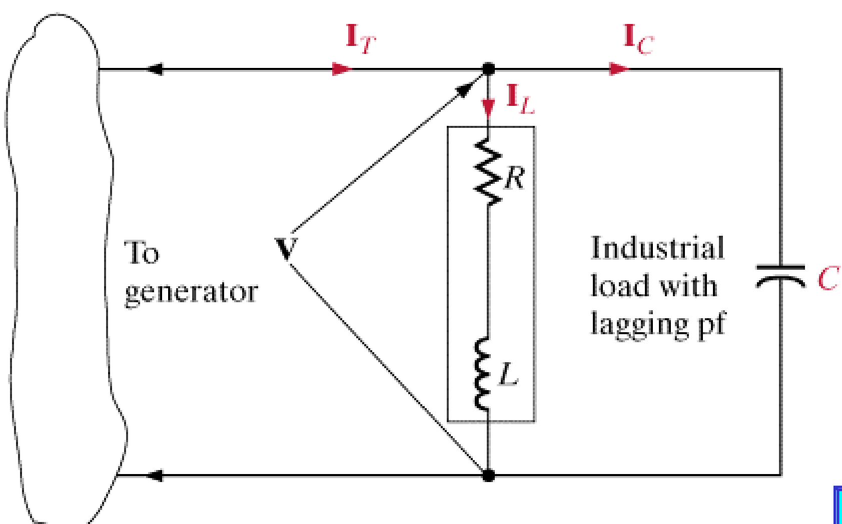
$$pf_{\text{new}} = \cos(\theta_{\text{new}})$$

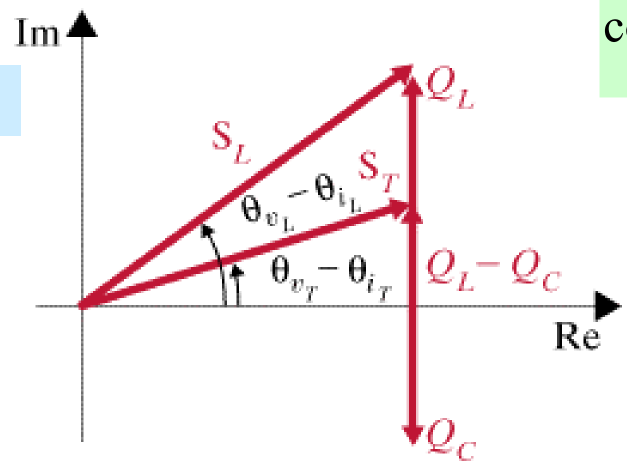
$$Q_{\text{capacitor}} = |V_L||I_{\text{capacitor}}| = |V_L|^2 \omega C$$

$$\tan\theta_{\text{new}} = \frac{Q_{\text{old}} - Q_{\text{capacitor}}}{P_{\text{old}}}$$

$$\cos\theta = \frac{1}{\sqrt{1 + \tan^2\theta}}$$

Simple approach to power factor correction



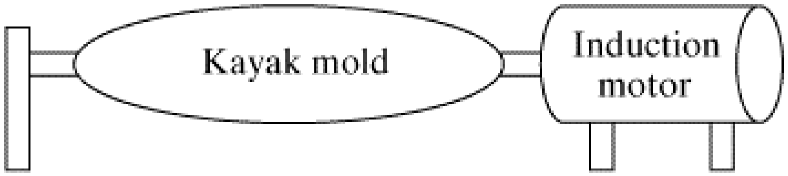


GEAUX

LEARNING EXAMPLE

$f = 60 Hz.$

Determine the capacitor required to increase the power factor to 0.95 lagging



Roto-molding process

$50kW, V_L = 220\angle 0°_{rms}$

$pf = 0.8\, lagging$

$$P = \text{Re}\{S\} = |S|\cos(\theta_v - \theta_i) = |S| \times pf$$

$$|S_{old}| = \frac{P}{pf} = \frac{50}{.80} = 62.5kVA \quad |Q_{old}| = \sqrt{|S_{old}|^2 - P^2} = 37.5(kVA).$$

$$\cos\theta_{new} = 0.95 \Rightarrow \tan\theta_{new} = \frac{\sqrt{1 - pf_{new}^2}}{pf_{new}} = 0.329 = \frac{Q_{new}}{P} \Rightarrow Q_{new} = 0.329 \times P = 16.43kVA$$

$$\therefore Q_{capacitor} = Q_{old} - Q_{new} = 37.5 - 16.43 = 21.07kVA$$

$$Q_{\text{capacitor}} = |V_L||I_{\text{capacitor}}| = |V_L|^2 \omega C$$

$$\therefore C = \frac{Q_{capacitor}}{\omega |V_L|^2} = \frac{21.07 \times 10^3}{(220)^2 \times (2\pi \times 60)} = 0.001156(F) = 1156\mu F$$

GEAUX

LEARNING EXTENSION

Determine the capacitor necessary to increase the power factor to 0.94

$$P_L = 100kW, V_L = 480(V)rms, pf = 0.707$$
$$R_{line} = 0.1\Omega, f = 60Hz$$

$$P = \text{Re}\{S\} = |S|\cos(\theta_v - \theta_i) = |S| \times pf$$

$$|S_{old}| = \frac{P}{pf} = \frac{100}{.707} = 141.44kVA \quad |Q_{old}| = \sqrt{|S_{old}|^2 - P^2} = 100.02(kVA).$$

$$\cos\theta_{new} = 0.94 \Rightarrow \tan\theta_{new} = \frac{\sqrt{1 - pf_{new}^2}}{pf_{new}} = 0.363 = \frac{Q_{new}}{P} \Rightarrow Q_{new} = 0.363 \times P = 36.3kVA$$

$$\therefore Q_{capacitor} = Q_{old} - Q_{new} = 100.02 - 36.3 = 63.72kVA$$

$$Q_{\text{capacitor}} = |V_L||I_{\text{capacitor}}| = |V_L|^2 \omega C$$

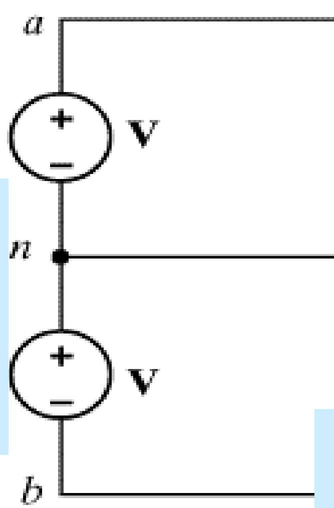
$$\therefore C = \frac{Q_{capacitor}}{\omega |V_L|^2} = \frac{63.72 \times 10^3}{(480)^2 \times (2\pi \times 60)} = 0.000733(F) = 733\mu F$$

GEAUX

SINGLE-PHASE THREE-WIRE CIRCUITS

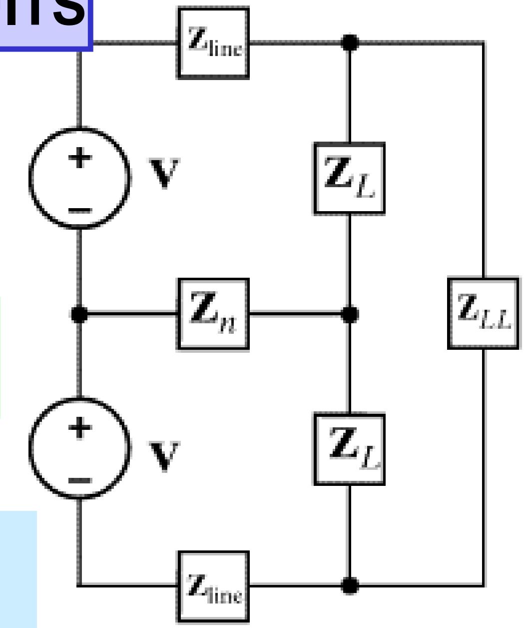
Power circuit normally used for residencial supply

Line-to-line used to supply major appliances (AC, dryer). Line-to-neutral for lights and small appliances



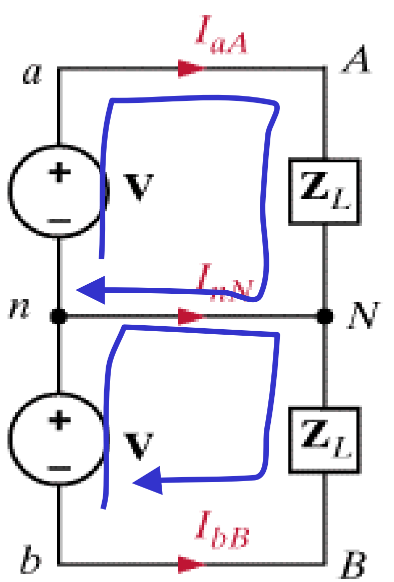
General balanced case

An exercise in symmetry

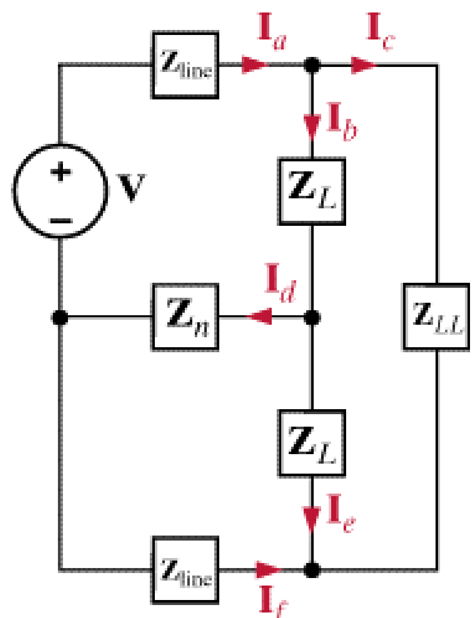


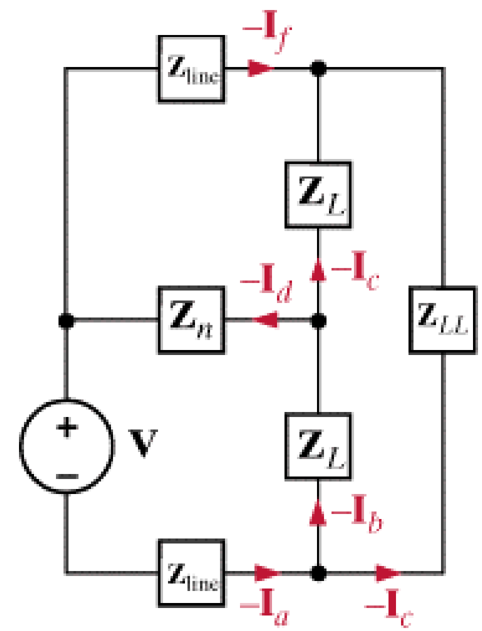
General case by source superposition

Basic circuit.



Neutral current is zero





Neutral current is zero

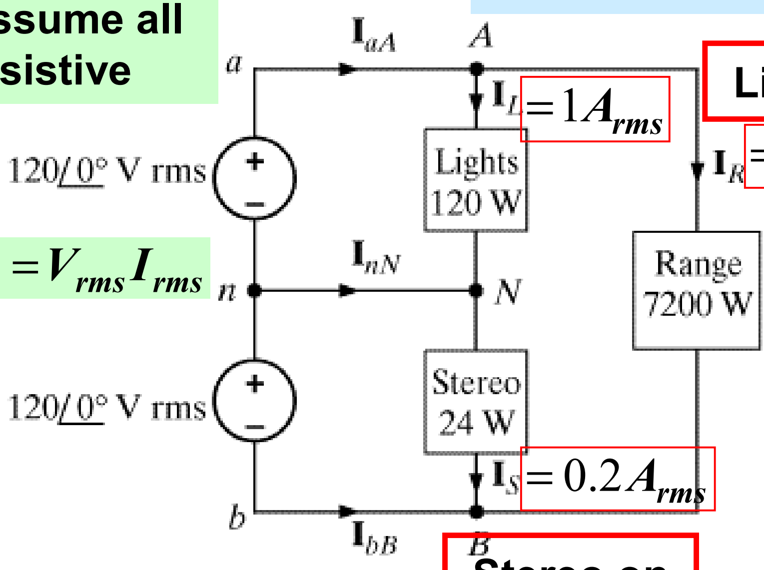
GEAUX

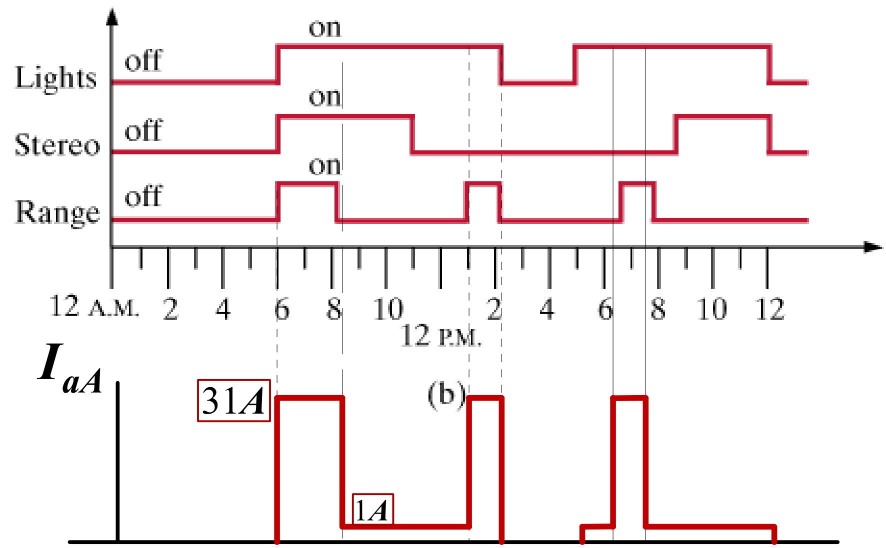
LEARNING EXAMPLE

Determine energy use over a 24-hour period and the cost if the rate is \$0.08/kWh

Assume all resistive

$P = V_{rms} I_{rms}$





$$Energy = \int p(t)dt = P_{average} \times Time$$

$$E_{lights} = 0.12kW \times 8Hr + 0.12kW \times 7Hr = 1.8kWh$$

$$E_{range} = 7.2kW \times (2+1+1)Hr = 28.8kWh$$

$$E_{stereo} = 0.024kW \times (5+3)Hr = 0.192kWh$$

$$E_{daily} = 30.792kWh$$

$$Cost = \$2.46 / day$$

KCL

$$I_{aA} = I_L + I_R$$

$$I_{bB} = -I_S - I_R$$

$$I_{nN} = I_S - I_L$$

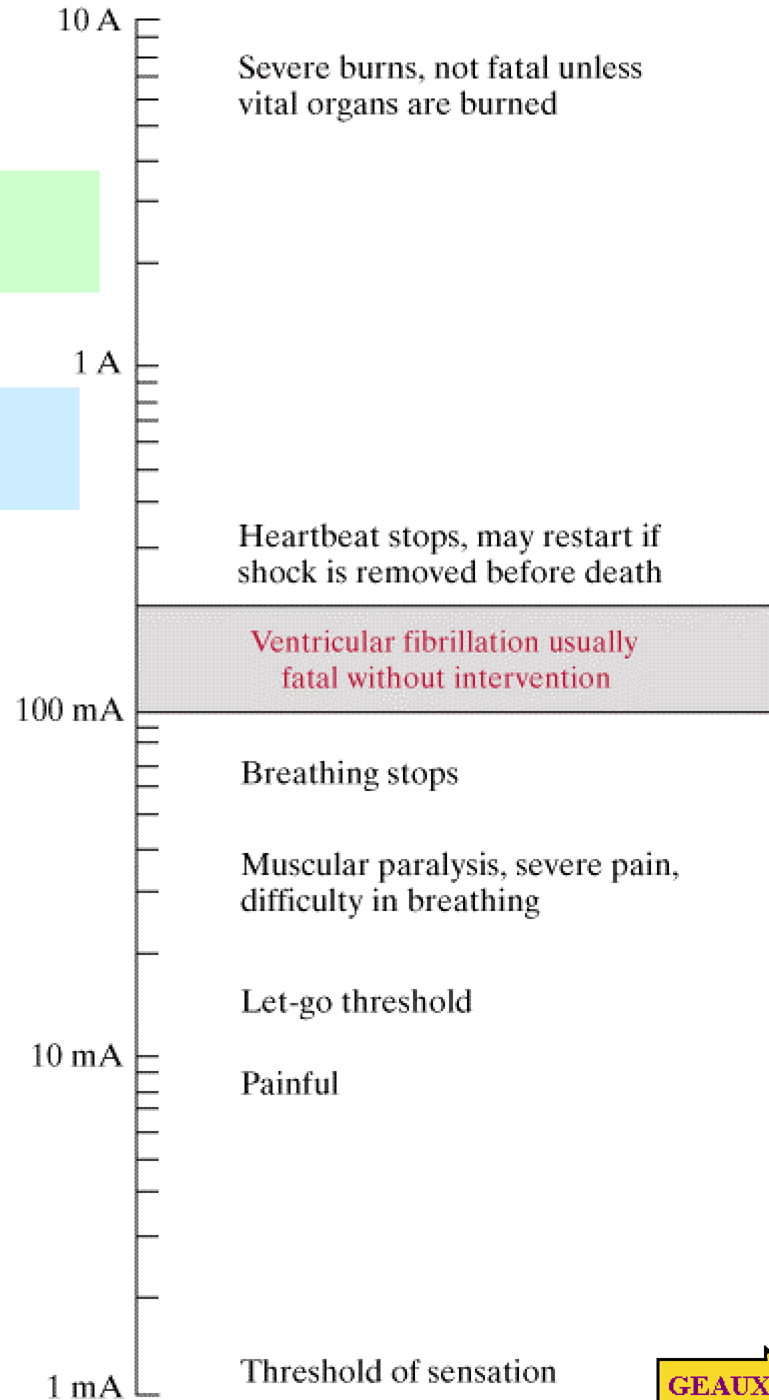
Outline of verification

$$E_{\text{supplied}} = \int p_{\text{supplied}} = V_{rms} \int I_{rms} dt$$

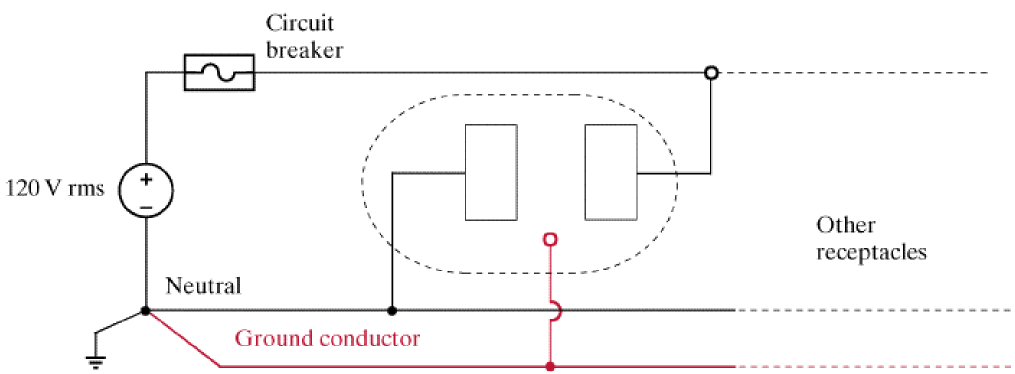
GEAUX

SAFETY CONSIDERATIONS

Average effect of 60Hz current from hand to hand and passing the heart



Required voltage depends on contact, person and other factors



Typical residential circuit with ground and neutral

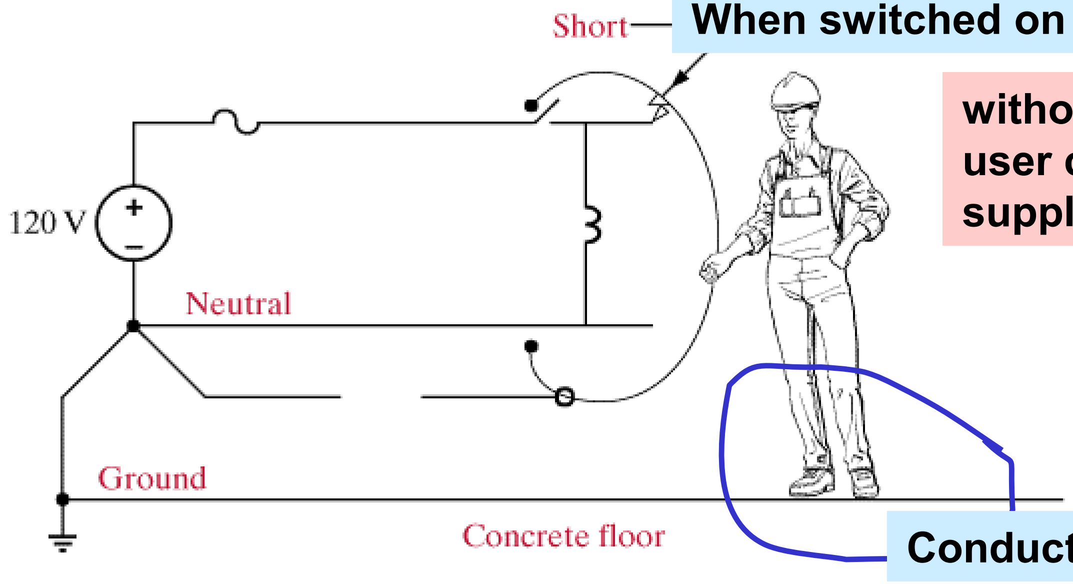
Ground conductor is not needed for normal operation

LEARNING EXAMPLE

Increased safety due to grounding

When switched on the tool case is energized

without the ground connector the user can be exposed to the full supply voltage!

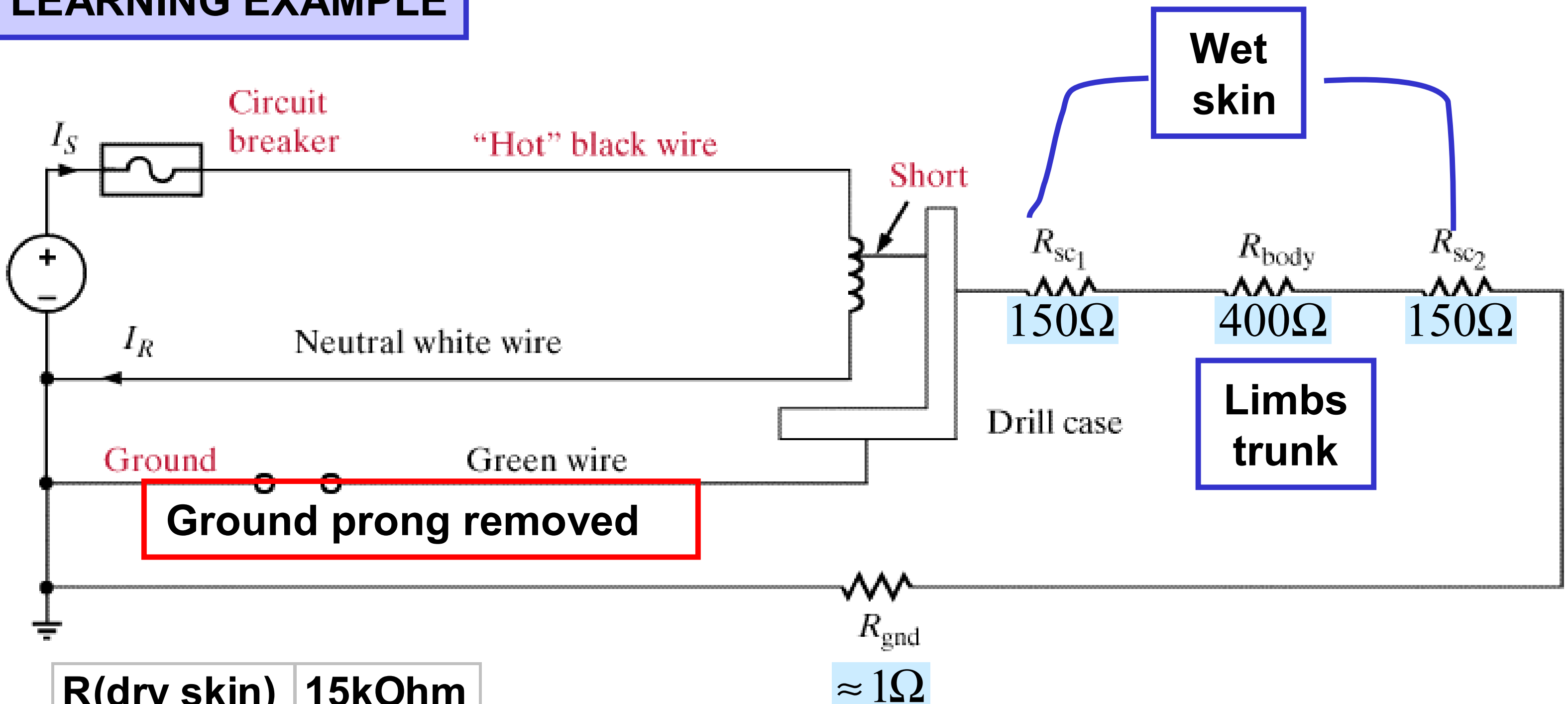


Conducting due to wet floor

If case is grounded then the supply is shorted and the fuse acts to open the circuit

More detailed numbers in a related case study

LEARNING EXAMPLE



R(dry skin)	15kOhm
R(wet skin)	150Ohm
R(limb)	100Ohm
R(trunk)	200Ohm

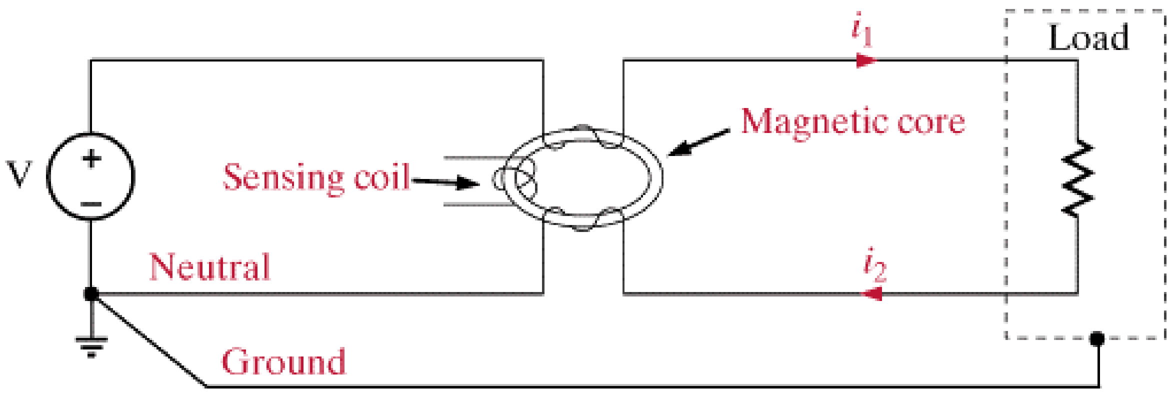
Suggested resistances for human body

$$I_{body} = \frac{120}{701} = 171 mA$$

Can cause ventricular fibrillation

LEARNING EXAMPLE

Ground Fault Interrupter (GFI)



In normal operating mode the two currents induce canceling magnetic fluxes

No voltage is induced in the sensing coil

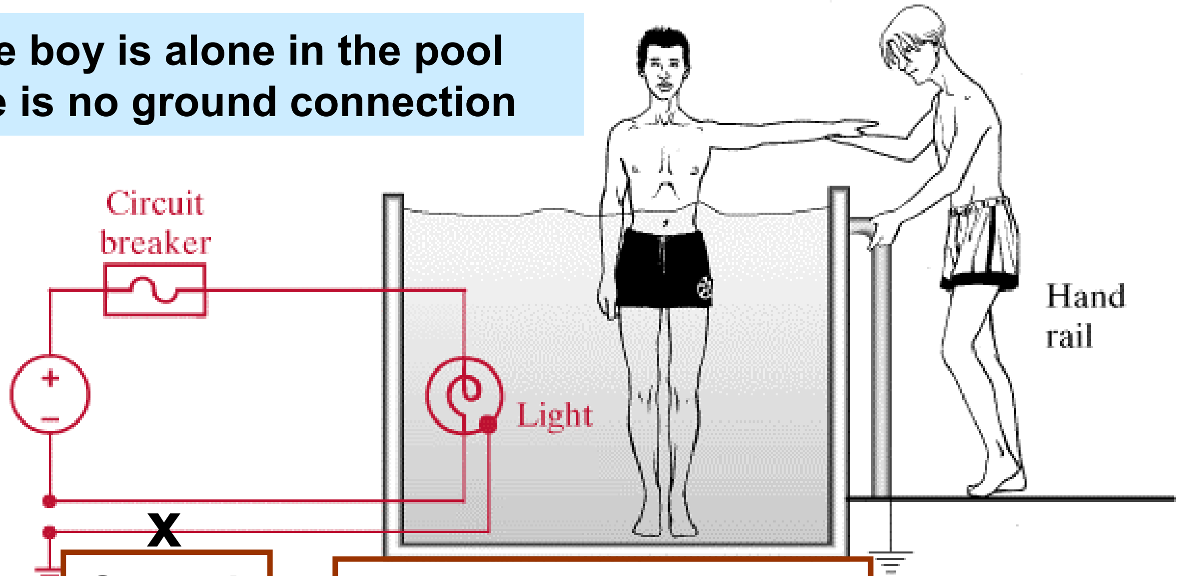
If i_1 and i_2 become different (e.g., due to a fault) then there is a voltage induced in the sensing coil

GEAUX

LEARNING EXAMPLE

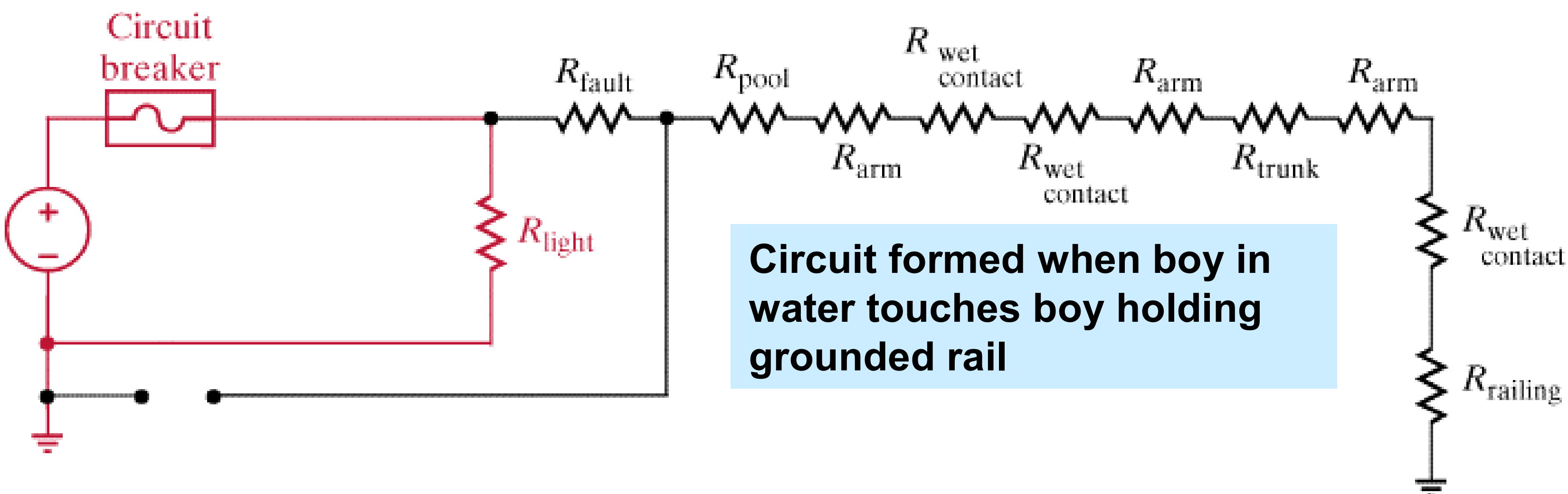
A ground fault scenario

While boy is alone in the pool there is no ground connection



Ground fault

Vinyl lining (insulator)

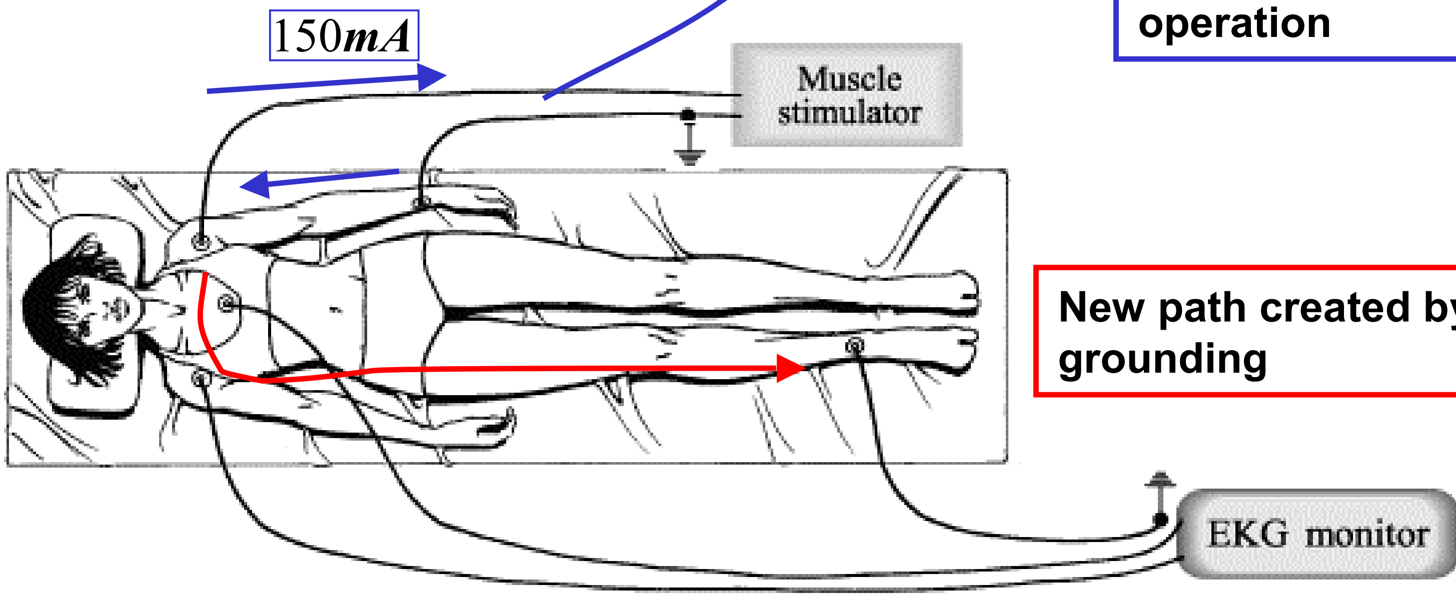


Circuit formed when boy in water touches boy holding grounded rail

GEAUX

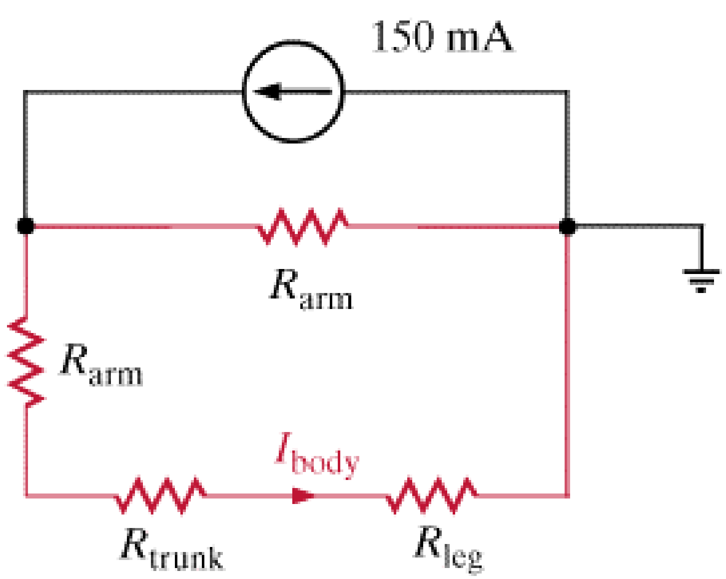
LEARNING EXAMPLE

Accidental grounding



Only return path in normal operation

New path created by the grounding

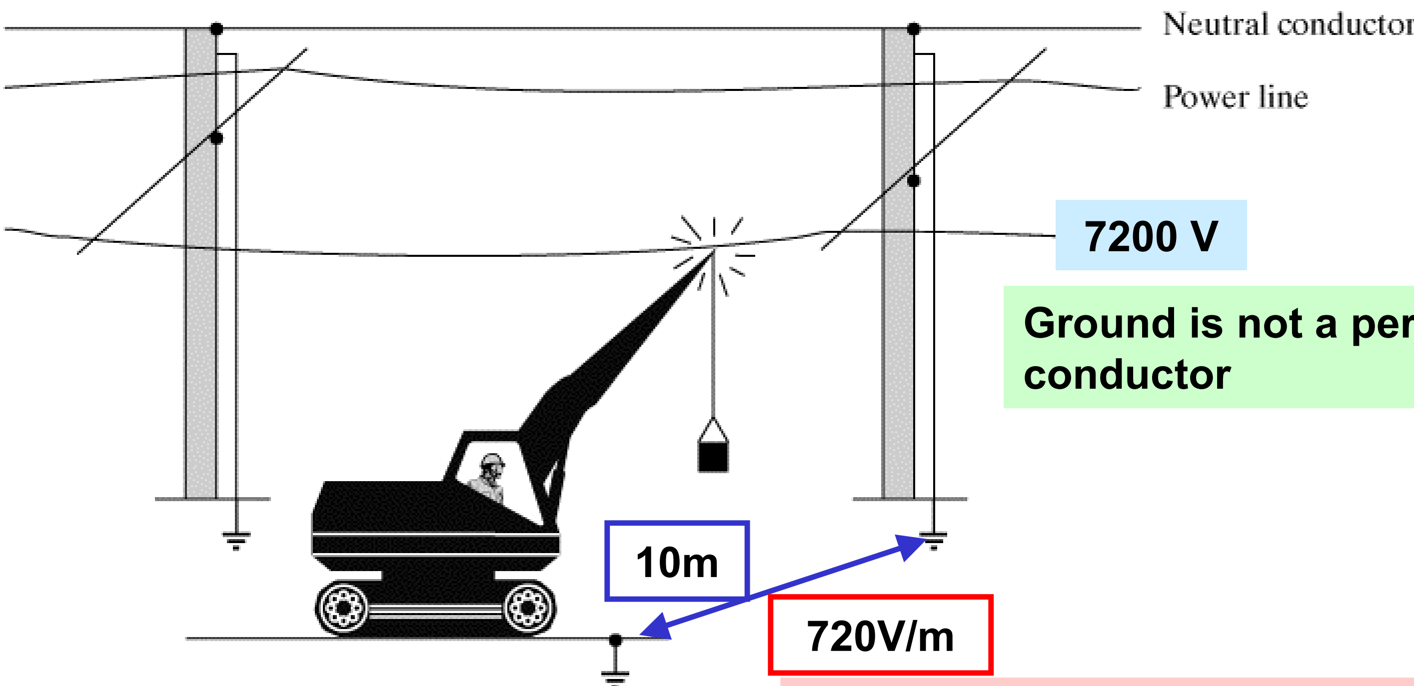


Using suggested values of resistance the secondary path causes a dangerous current to flow through the body

LEARNING EXAMPLE

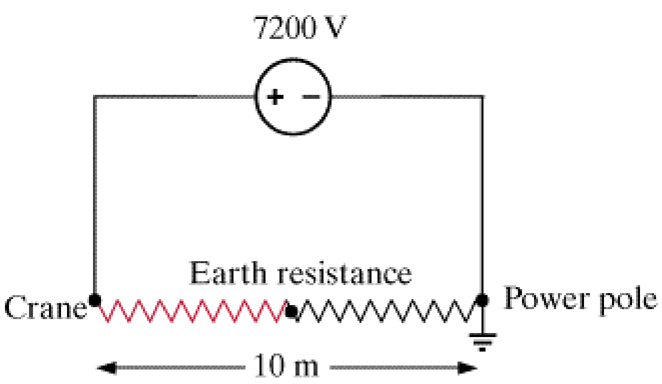
A grounding accident

After the boom touches the live line the operator jumps down and starts walking towards the pole



Ground is not a perfect conductor

One step applies 720 Volts to the operator

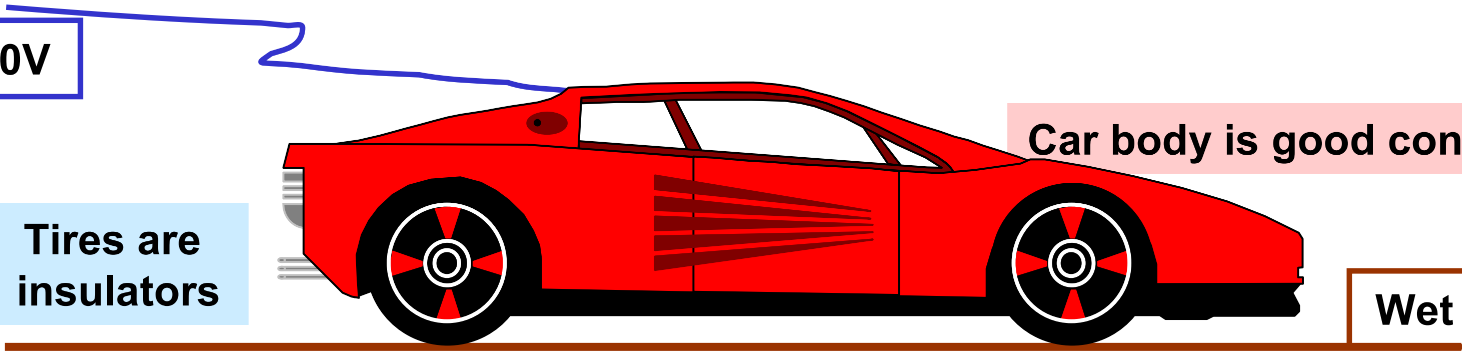


(b)

GEAUX

LEARNING EXAMPLE

A 7200V power line falls on the car and makes contact with it



Option 1.
Driver opens door and steps down

Option 2:
Driver stays inside the car

R(dry skin)	15kOhm
R(wet skin)	150Ohm
R(limb)	100Ohm
R(trunk)	200Ohm

Suggested resistances for human body

$$I_{\text{body}} = \frac{7200}{R_{\text{dry skin}} + 2R_{\text{limb}} + R_{\text{trunk}}}$$

$I \approx 460mA$ Very dangerous!

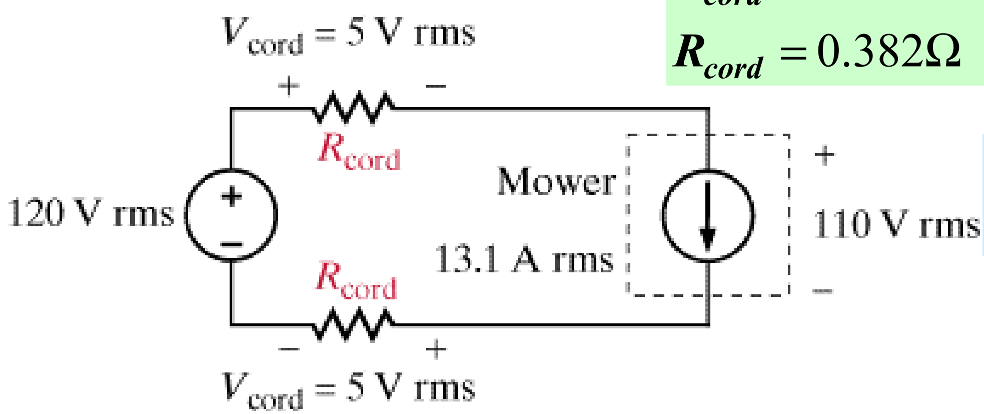
$$I_{\text{body}} = 0$$

LEARNING EXAMPLE

Find the maximum cord length

$$R_{cord} \times 13.1A = 5V$$

$$R_{cord} = 0.382\Omega$$



Minimum voltage for proper operation

CASE 1: 16-gauge wire $4\frac{m\Omega}{ft}$

$$L = \frac{R_{cord}}{4\, {}^{m\Omega}/_{ft}} = 95.5\, ft$$

CASE 2: 14-gauge wire $2.5\frac{m\Omega}{ft}$

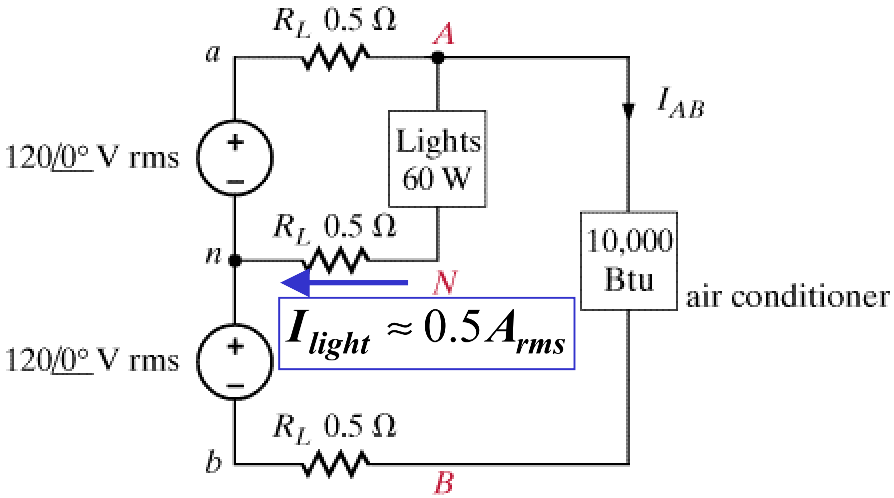
$$L = \frac{R_{cord}}{2.5\, {}^{m\Omega}/_{ft}} = 152.8\, ft$$

Working with RMS values the problem is formally the same as a DC problem

GEAUX

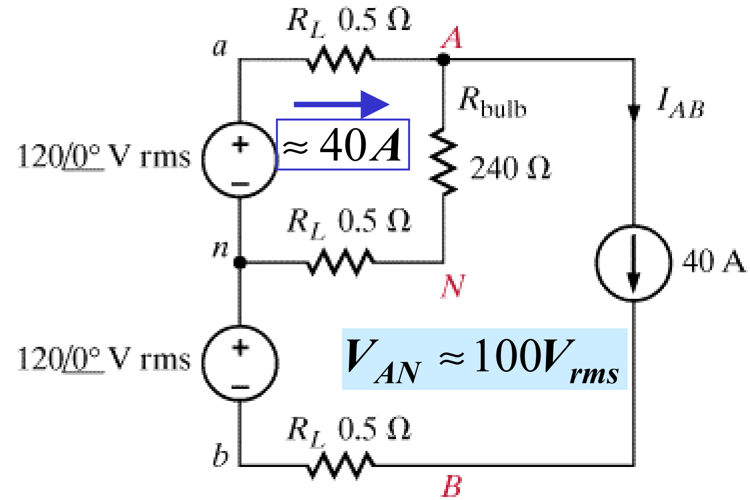
LEARNING EXAMPLE

Light dimming when AC starts



$$I_{light} \approx 0.5 A_{rms}$$

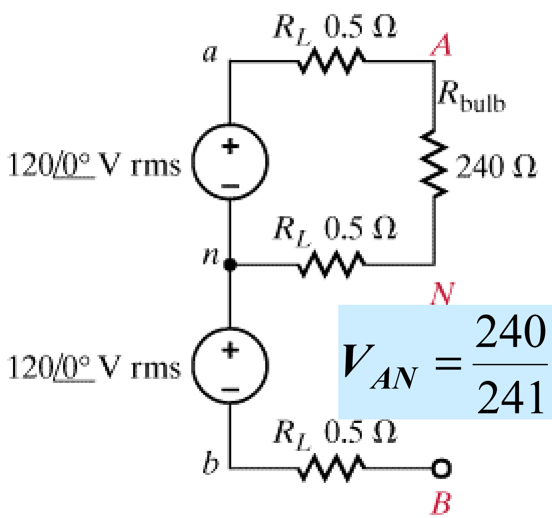
Typical single-phase 3-wire installation



$$\approx 40A$$

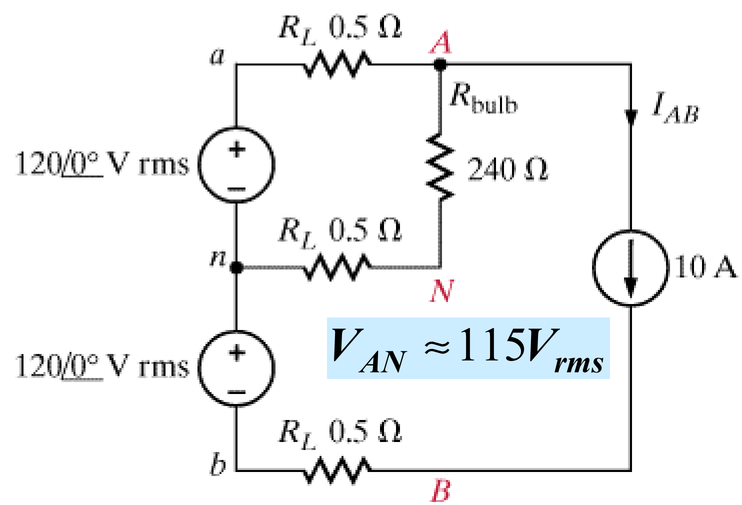
$$V_{AN} \approx 100 V_{rms}$$

Circuit at start of AC unit. Current demand is very high



$$V_{AN} = \frac{240}{241} \times 120 = 119.5 V_{rms}$$

AC off



$$V_{AN} \approx 115 V_{rms}$$

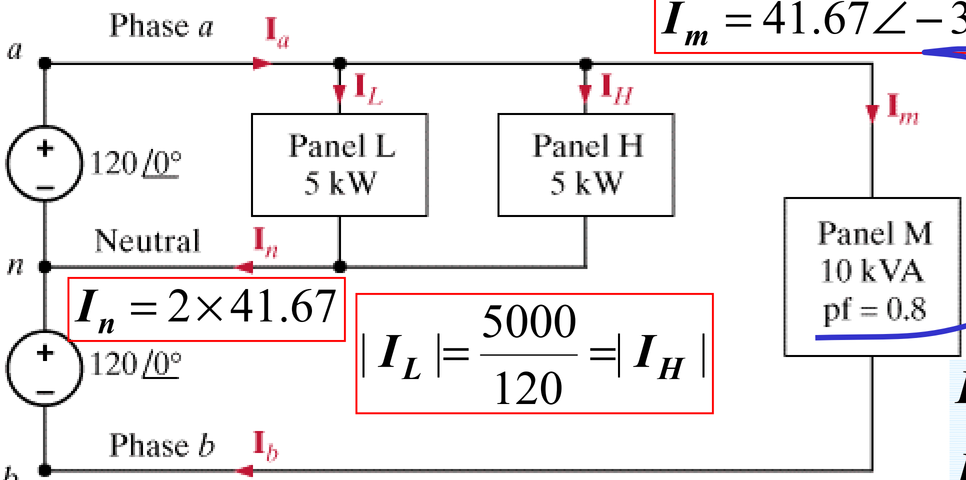
AC in normal operation

LEARNING BY DESIGN

Analysis of single phase 3-wire circuit installations

$$I_a = 41.67\angle 0° + 41.67\angle 0° + 41.67\angle -36.9° = 119.4\angle -12.1° (A)$$

$$I_m = 41.67\angle -36.9°$$



$$I_n = 2 \times 41.67$$

$$|I_L| = \frac{5000}{120} = |I_H|$$

$$I_b = I_m$$

$$S_A = 120\angle 0° \times [119.4\angle -12.1°]^* = 14.38\angle 12.1° kVA = 14 + j3kVA$$

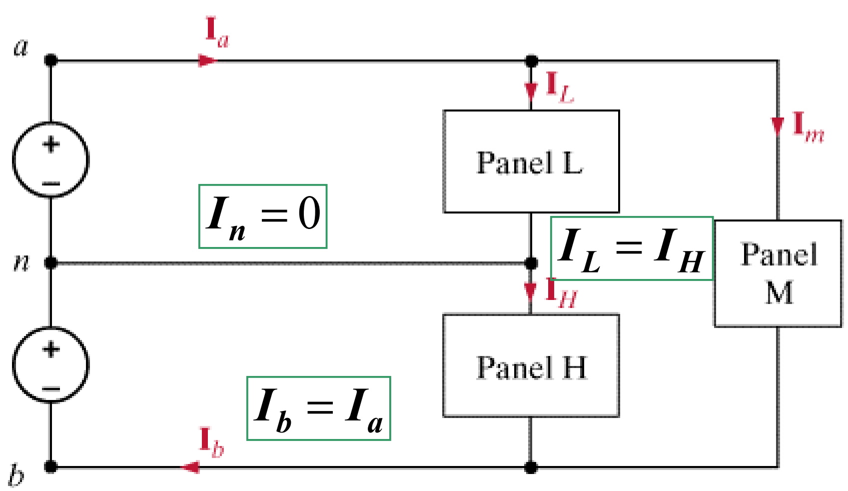
$$S_B = 120\angle 0° \times [41.67\angle -36.9°]^* = 5\angle 36.9° kVA = 4 + j3kVA$$

$$R_{lines} = 0.05\Omega$$

$$P_{losses} = 0.05 \times \left(|I_a|^2 + |I_b|^2 + |I_n|^2\right) = 1.147kW$$

Option 1

$$I_a = 41.67\angle 0° + 41.67\angle -36.9° = 79.07\angle -18.4° (A)$$



$$I_n = 0$$

$$I_L = I_H$$

$$I_b = I_a$$

$$S_A = S_B = 120\angle 0° \times [79.07\angle 18.4°]^* = 9.5\angle 18.4 kVA° = 9 + j3kVA$$

$$R_{lines} = 0.05\Omega$$

$$P_{losses} = 0.05 \times \left(|I_a|^2 + |I_b|^2\right) = 0.625kW$$

$$P_{saved} = 0.522kW$$

$$\$/year = 366 (@0.08\$/kWh)$$

Option 2

Steady-state Power Analysis

STOP