Cosine-Modulated *L*-Channel Filter Banks

- Originally developed to provide nearly perfect reconstruction with aliasing cancellation between adjacent channels and assume no aliasing between nonadjacent channels due to infinite stopband attenuation of the analysis filters in all nonadjacent bands
- The filter banks are also known as pseudo-QMF banks

Cosine-Modulated L-Channel Filter Banks

- Note: The assumption of no aliasing between nonadjacent channels do not hold in practice
- However, these filter banks provide quite satisfactory performance if the stopband attenuation is sufficiently high
- The pseudo-QMF banks are derived from a modified form of the uniform DFT filter banks

Cosine-Modulated *L*-Channel Filter Banks

- Let

$$P_0(z) = \sum_{n=0}^{N} p_0[n] z^{-n}$$

denote the prototype lowpass filter with real coefficients and a cutoff at $\pi/2L$

- We generate a set of filters $Q_k(z)$ from $P_0(z)$ by complex modulation at frequencies $(2k+1)\pi/2L = (k+0.5)\pi/L$ as follows:

$$Q_k(z) = P_0(zW_{2L}^{k+0.5}), \qquad 0 \le k \le 2L-1$$

Cosine-Modulated *L*-Channel Filter Banks

- Note: $W_{2L} = e^{-j\pi/L}$
- Because of the complex modulation, the filters $Q_k(z)$ have complex-valued impulse responses
- Note: The response of $Q_0(z)$ is a right-shifted version of the response of $P_0(z)$ shifted by $\pi/2L$

Cosine-Modulated *L*-Channel Filter Banks

- Because of the shift, $|Q_k(e^{j\omega})| = |Q_{2L-1-k}(e^{j\omega})|$ and the impulse response of $Q_{2L-1-k}(z)$ is complex conjugate of the impulse response of $Q_k(z)$, for $0 \le k \le 2L-1$
- The pair $Q_k(z)$ and $Q_{2L-1-k}(z)$ are combined to generate a filter with real impulse response

Cosine-Modulated *L*-Channel Filter Banks

- Define the intermediate transfer functions

$$U_k(z) = c_k Q_k(z), \quad V_k(z) = c_k^* Q_{2L-1-k}(z)$$

- The L analysis filters are then formed according to

$$H_k(z) = \sum_{n=0}^{N} h_k[n] z^{-n}$$

$$= a_k U_k(z) + a_k^* V_k(z), \quad 0 \le k \le L-1$$

Cosine-Modulated *L*-Channel Filter Banks

- Likewise, the L synthesis filters are formed according to

$$G_k(z) = \sum_{n=0}^{N} g_k[n] z^{-n}$$

$$= b_k U_k(z) + b_k^* V_k(z), \quad 0 \le k \le L-1$$

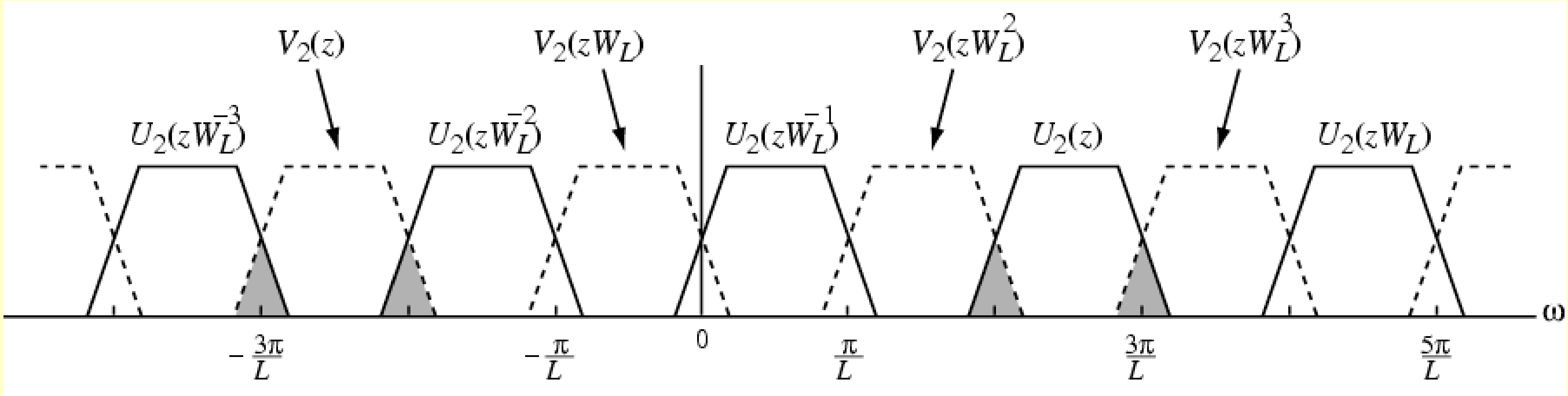
- In the above equations, a_k , b_k , and c_k are unit-magnitude constants

Cosine-Modulated L-Channel Filter Banks

- These 3 constants are chosen to provide alias cancellation between adjacent channels and to ensure that the distortion transfer function has linear phase
- Consider the 2nd channel
- The output of the filter $G_2(z)$ has the components $H_2(zW_L^{\ell})X(zW_L^{\ell})$, $0 \le \ell \le L-1$

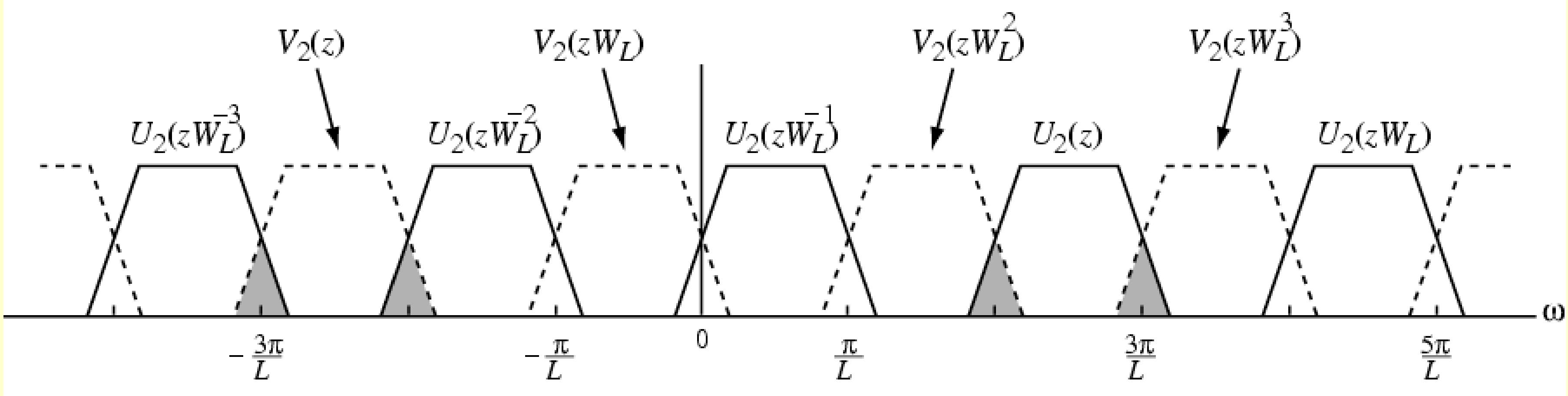
Cosine-Modulated *L*-Channel Filter Banks

- However, as can be seen from the figure given below, the responses of $U_2(zW_L)$ and $U_2(zW_L^{-1})$ do not overlap with that of $U_2(z)$



Cosine-Modulated *L*-Channel Filter Banks

- On the other hand, the responses of $U_2(zW_L^{-2})$ and $U_2(zW_L^{-3})$ overlap with that of $V_2(z)$



Cosine-Modulated *L*-Channel Filter Banks

- In general, "significant" alias components of $X(zW_L^{\ell})$ at the output of $G_k(z)$ correspond to values of

$$\ell = \langle -k+1\rangle_L, \langle -k\rangle_L, k, k+1$$

- Similarly, significant alias components of $X(zW_L^{\ell})$ at the output of $G_{k-1}(z)$ correspond to values of

$$\ell = \langle -k\rangle_L, \langle -k+1\rangle_L, k-1, k$$

Cosine-Modulated L-Channel Filter Banks

- Additional requirement: The pseudo-QMF bank should have linear phase
- To this end, the distortion function $T(z)$ as given by

$$T(z) = a_0(z) = \frac{1}{L}\sum_{k=0}^{L-1} H_k(z) G_k(z)$$

  should have linear phase
- This is achieved if $G_k(z) = z^{-N} H_k(z^{-1})$

Cosine-Modulated L-Channel Filter Banks

- It can be shown that the constants a_k, b_k, and c_k, can be chosen to cancel the common aliasing components $X(zW_L^{\pm k})$ at the outputs of $G_k(z)$ and $G_{k-1}(z)$, and to make $T(z)$ have linear phase
- These values of the constants also lead to closed-form expressions for the analysis and synthesis filters

Cosine-Modulated L-Channel Filter Banks

- The expressions for the analysis and synthesis filters are related to the prototype filter $p_0[n]$ by cosine modulation:

$$h_k[n] = 2\,p_0[n]\cos\left[(k+\tfrac{1}{2})(n-\tfrac{N}{2})\tfrac{\pi}{L} + (-1)^k \tfrac{\pi}{4}\right]$$

$$g_k[n] = 2\,p_0[n]\cos\left[(k+\tfrac{1}{2})(n-\tfrac{N}{2})\tfrac{\pi}{L} - (-1)^k \tfrac{\pi}{4}\right]$$

- Note: If the prototype filter $P_0(z)$ has linear phase, the distortion transfer function $T(z)$ has linear phase

Prototype Lowpass Filter Design

- The design of the cosine-modulated filter bank reduces to the design of the L-th band prototype lowpass filter $p_0[n]$ such that the magnitude response $|T(e^{j\omega})|$ of the distortion transfer function $T(z)$ is approximately flat for all values of ω
- To this end, $P_0(z)$ should satisfy as much as possible the following 2 conditions:

Prototype Lowpass Filter Design

(1) $|P_0(e^{j\omega})|^2 + |P_0(e^{j(\omega-\pi/L)})|^2 = 1, \quad 0 < \omega < \frac{\pi}{L}$

(2) $|P_0(e^{j\omega})| = 0, \quad \omega > \frac{\pi}{L}$

- The QMF bank does not exhibit any amplitude distortion if if the top equation is satisfied exactly
- There is no aliasing between nonadjacent channels if the bottom equation holds

Prototype Lowpass Filter Design

- As indicated earlier, aliasing between adjacent channels is cancelled structurally
- The design of an L-th band FIR prototype filter satisfying both conditions is not posible
- A relatively straightforward design approach makes use of the popular Parks-McClellan method to design the prototype lowpass filter

Prototype Lowpass Filter Design

- The two conditions are satisfied approximately by adjusting iteratively the passband edge to minimize the objective function

$$\phi = \max_{0<\omega<\pi/L} \{|P_0(e^{j\omega})|^2 + |P_0(e^{j(\omega-\pi/L)})|^2 - 1\}$$

- The filter length, stopband edge at π/L, and the relative error weighting are kept fixed during the optimization procedure

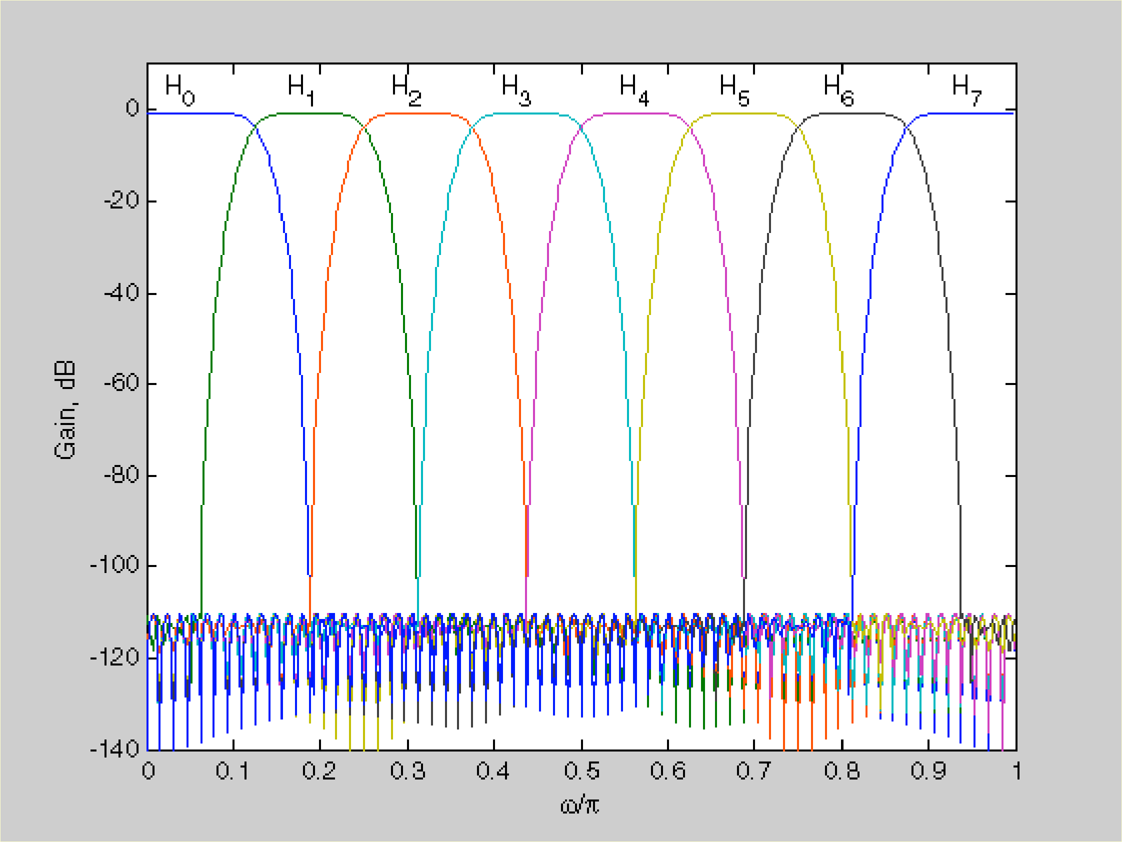
Cosine-Modulated *L*-Channel Filter Bank Design Example

- Example - Design an eighth-band pseudo QMF analysis bank using a prototype eighth-band lowpass FIR filter of length 128
- Using the function `opt_filter` we first designed the prototype eighth-band lowpass FIR filter

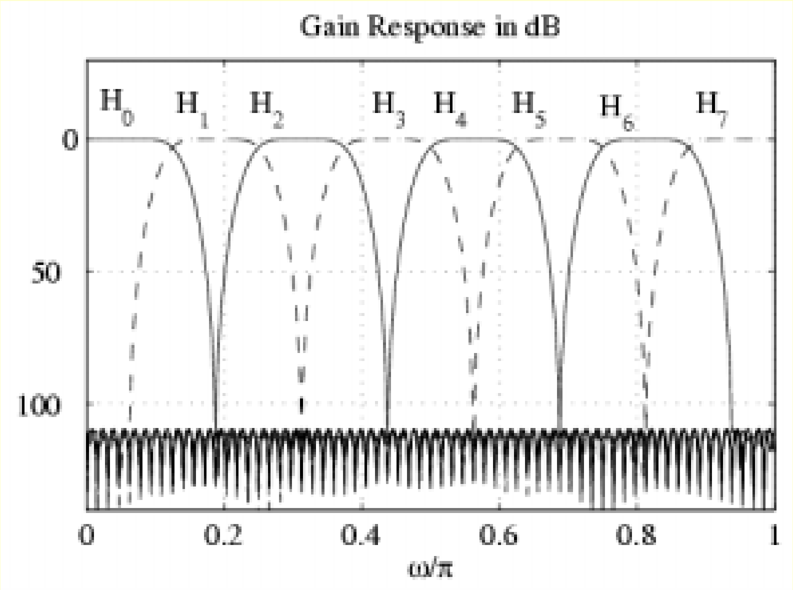
Cosine-Modulated *L*-Channel Filter Bank Design Example

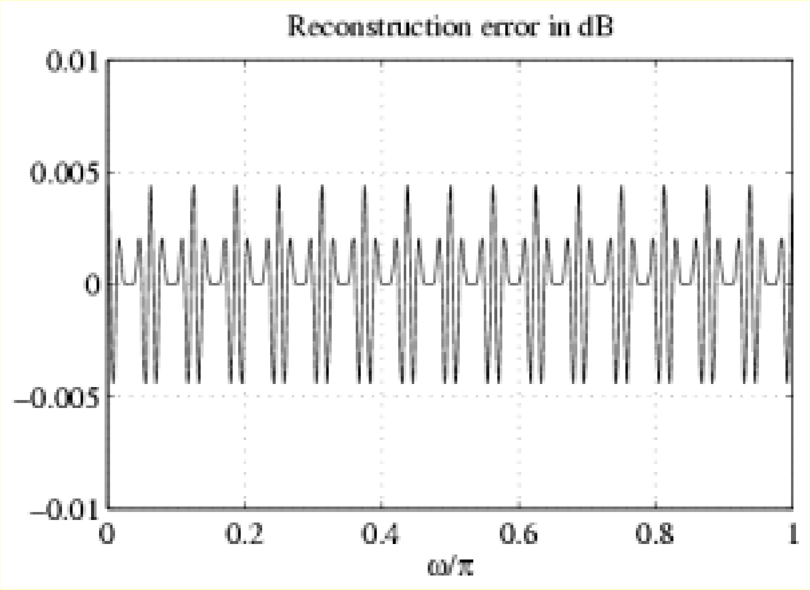
- Next, using the function `make_bank`, the coefficients of the remaining analysis filters, and the 8 synthesis filters are determined
- Figures on the next slide show the gain responses of the 8 analysis filters $H_k(z)$ and the reconstruction error in dB

Cosine-Modulated *L*-Channel Filter Bank Design Example



Cosine-Modulated *L*-Channel Filter Bank Design Example





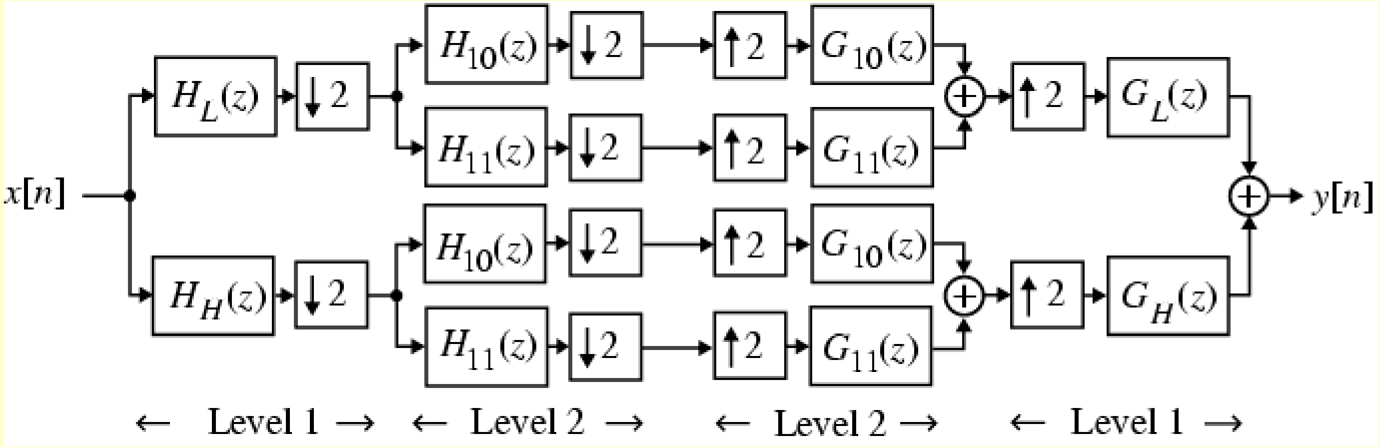
Multilevel Filter Banks

- Multiband analysis/synthesis filter banks can also be designed by iterating a 2-channel QMF bank
- Moreover, if the 2-channel QMF bank is of perfect reconstruction type, the generated multiband structure also exhibits perfect reconstruction property

Multilevel Filter Banks with Equal Passband Widths

- A 4-channel maximally decimated QMF bank can be designed by inserting a 2-channel maximally decimated QMF bank in each channel of another maximally decimated QMF bank as shown on the next slide

Multilevel Filter Banks with Equal Passband Widths



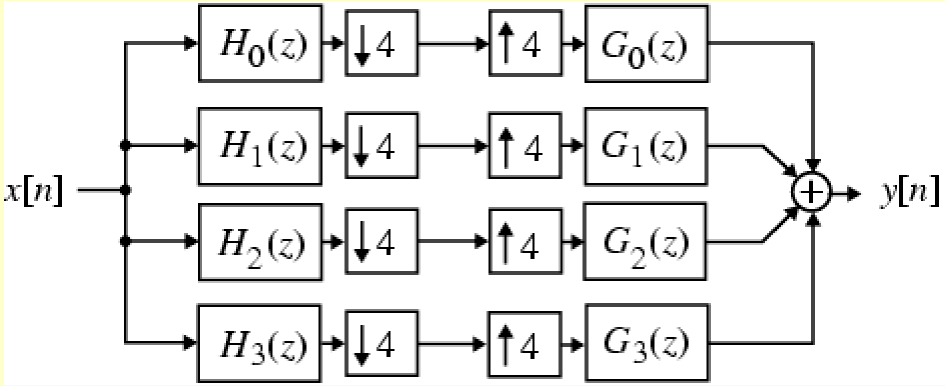
- Since the analysis and synthesis filter banks are formed like a tree, the overall system is often called a tree-structured filter bank

Multilevel Filter Banks with Equal Passband Widths

- In the 4-channel tree-structured filter bank shown on the previous slide, the 2-channel QMF banks in the second level do not have to be identical
- If they are not identical, to compensate for the unequal gains and unequal delays of the 2-channel systems, additional delays of appropriate values need to be inserted to ensure perfect reconstruction

Multilevel Filter Banks with Equal Passband Widths

- An equivalent representation of the 4-channel tree-structured filter banks is shown below



Multilevel Filter Banks with Equal Passband Widths

- The analysis filters in the equivalent representation are related to those of the parent 2-level QMF bank as follows:

$$H_0(z) = H_L(z)H_{10}(z^2)$$

$$H_1(z) = H_L(z)H_{11}(z^2)$$

$$H_2(z) = H_H(z)H_{10}(z^2)$$

$$H_3(z) = H_H(z)H_{11}(z^2)$$

Multilevel Filter Banks with Equal Passband Widths

- Likewise, the synthesis filters in the equivalent representation are related to those of the parent 2-level QMF bank as follows:

$$G_0(z) = G_L(z)G_{10}(z^2)$$

$$G_1(z) = G_L(z)G_{11}(z^2)$$

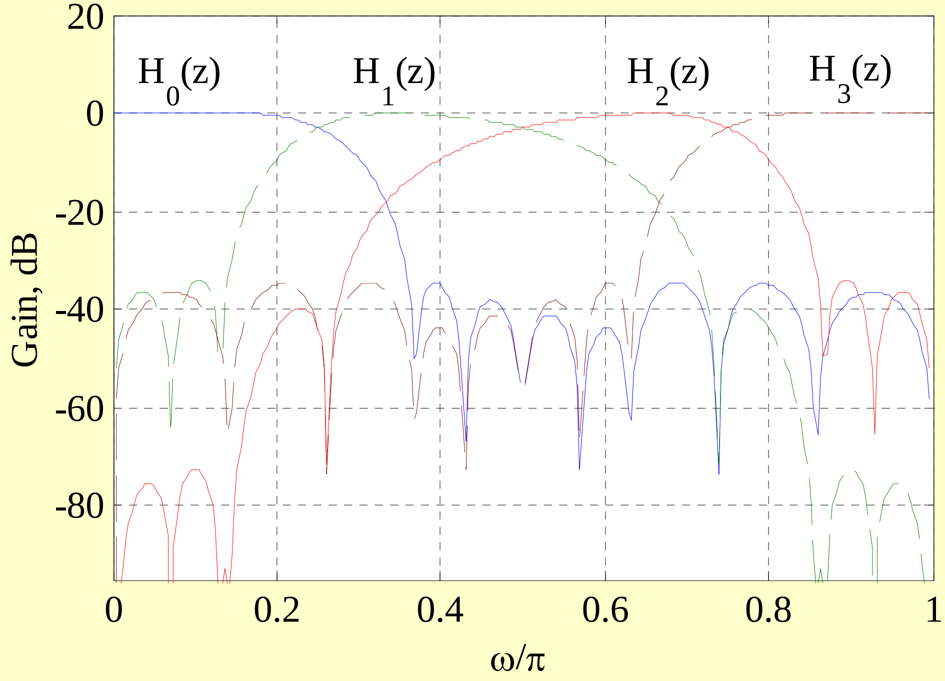
$$G_2(z) = G_H(z)G_{10}(z^2)$$

$$G_3(z) = G_H(z)G_{11}(z^2)$$

Multilevel Filter Banks with Equal Passband Widths

- Example - We design a 4-channel QMF bank by iterating the 2-channel QMF bank based on the filter 12B of Johnston
- Using Program 10_10 we compute the impulse response of the 4 analysis filters and then determine their gain responses as shown on the next slide

Multilevel Filter Banks with Equal Passband Widths



Multilevel Filter Banks with Equal Passband Widths

- Each analysis filter $H_k(z)$ in the equivalent representation is essentially a cascade of two filters, one with a single passband and a single stopband and the other with two passbands and two stopbands
- The passband of the cascade is the frequency range where the passbands of the two filters overlap

Multilevel Filter Banks with Equal Passband Widths

- On the other hand, the stopband of the cascade is formed from three different frequency ranges
- In two of the frequency ranges, the passband of one coincides with the stopband of the other, while in the third range, the two stopbands overlap

Multilevel Filter Banks with Equal Passband Widths

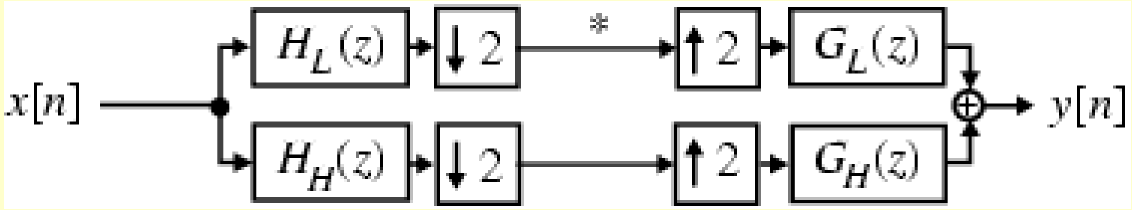
- As a result, the gain responses of the cascade in the three regions are not equal, resulting in an uneven attenuation characteristics
- This type of behavior of the gain response can be seen in the plots shown earlier and should be taken into account in the design of the tree-structured filter bank

Multilevel Filter Banks with Equal Passband Widths

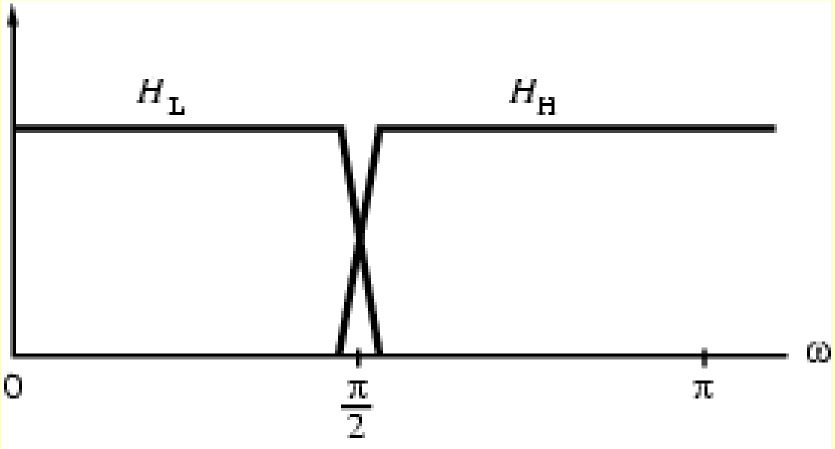
- By continuing the process, QMF banks with more than 4 channels can be easily constructed
- The number of channels resulting from this approach is restricted to be a power of 2, i.e., $L = 2^{\nu}$
- Also, the filters in the analysis (synthesis) branch have passbands of equal width π/L

Filter Banks with Unequal Passband Widths

- Consider the 2-channel maximally decimated QMF bank shown below

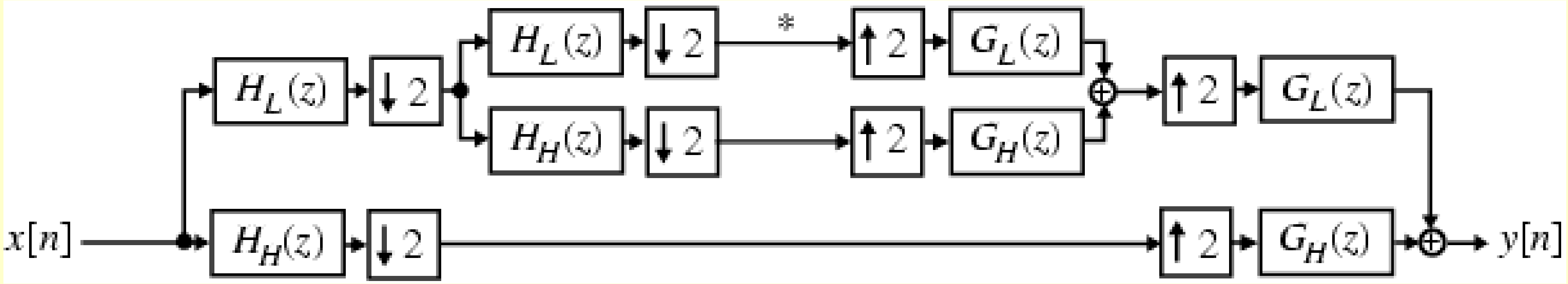


with a typical magnitude response



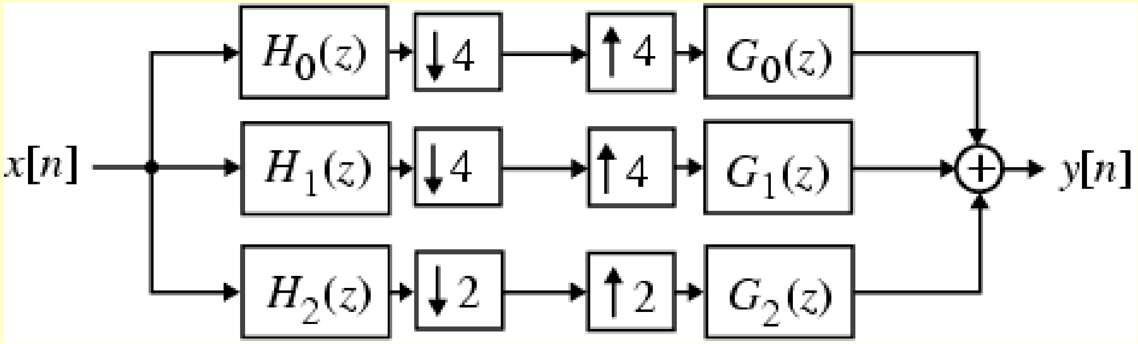
Filter Banks with Unequal Passband Widths

- By inserting another 2-channel maximally decimated QMF bank in the top subband channel at the position marked by a * we arrive at a 3-channel maximally decimated QMF bank as shown below



Filter Banks with Unequal Passband Widths

- The equivalent representation of the generated 3-channel filter bank is shown below



Filter Banks with Unequal Passband Widths

- The analysis and synthesis filters here are given by

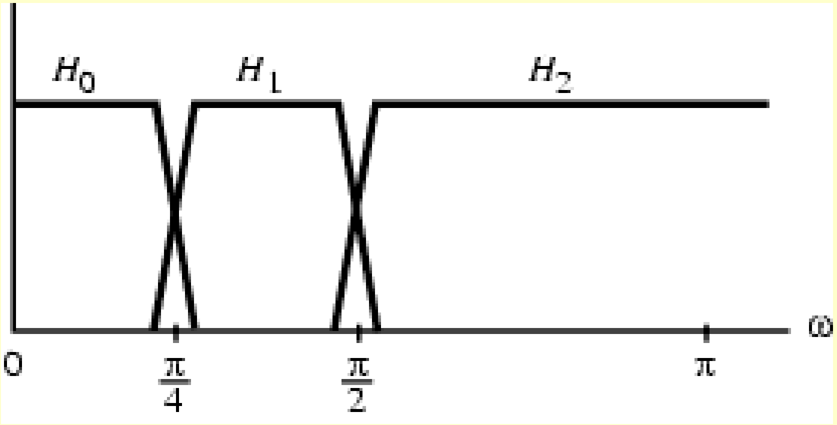
$$H_0(z) = H_L(z)H_L(z^2), \quad G_0(z) = G_L(z)G_L(z^2)$$

$$H_1(z) = H_L(z)H_H(z^2), \quad G_1(z) = G_L(z)G_H(z^2)$$

$$H_2(z) = H_H(z), \quad G_2(z) = G_H(z)$$

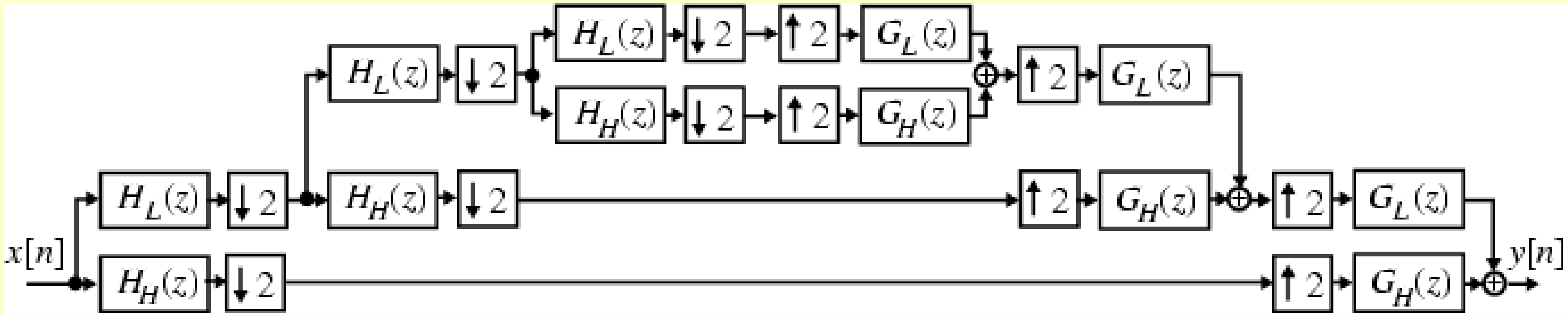
Filter Banks with Unequal Passband Widths

- Typical magnitude responses of the analysis filters of the derived 3-channel QMF bank are shown below



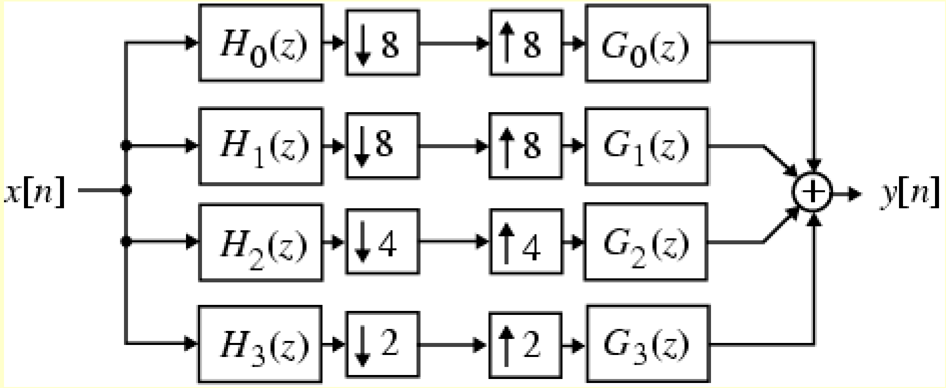
Filter Banks with Unequal Passband Widths

- We can continue this process and generate a 4-channel QMF bank from the 3-channel QMF bank by inserting a 2-channel QMF bank in the top subband channel marked with a *



Filter Banks with Unequal Passband Widths

- Its equivalent 4-channel representation is shown below



Filter Banks with Unequal Passband Widths

- The analysis filters here are given by

$$H_0(z) = H_L(z)H_L(z^2)H_L(z^4)$$

$$H_1(z) = H_L(z)H_L(z^2)H_H(z^4)$$

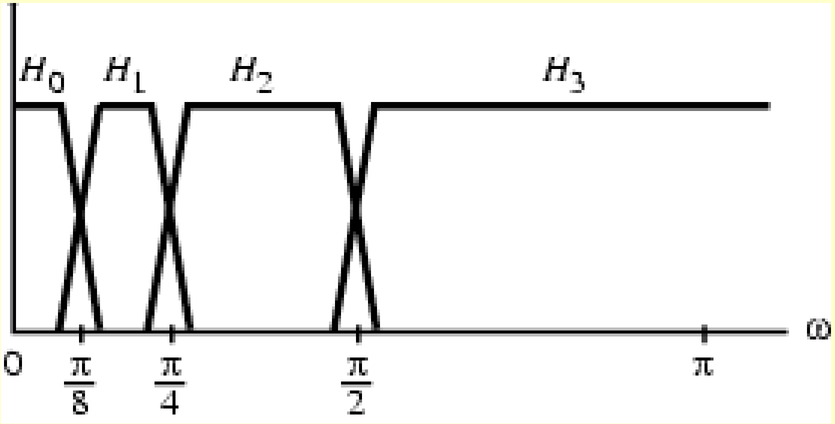
$$H_2(z) = H_L(z)H_H(z^2)$$

$$H_3(z) = H_H(z)$$

- Corresponding expressions for the synthesis filters can be derived

Filter Banks with Unequal Passband Widths

- Figure below shows typical magnitude responses of the 4-channel QMF bank derived from a parent 2-channel QMF bank



Filter Banks with Unequal Passband Widths

- Because of the unequal passband widths of the analysis and synthesis filters, these QMF structures belong to the class of nonuniform QM banks
- The tree-structured filter banks have also been referred to as the octave band QMF banks