Chapter 5

5.1 Let $\tilde{y}[n] = \sum_{r=0}^{N-1} \tilde{x}[r]\tilde{h}[n-r]$. Then $\tilde{y}[n+kN] = \sum_{r=0}^{N-1} \tilde{x}[r]\tilde{h}[n+kN-r]$. Since $\tilde{h}[n]$ is periodic in n with a period $N, \tilde{h}[n+kN-r] = \tilde{h}[n-r]$. Therefore $\tilde{y}[n+kN]$ $= \sum_{r=0}^{N-1} \tilde{x}[r]\tilde{h}[n-r] = \tilde{y}[n]$, hence $\tilde{y}[n]$ is also periodic in n with a period N.

5.2 **(a)** $\tilde{y}[0] = \sum_{r=0}^{4} \tilde{x}[r]\tilde{h}[-r] = \tilde{x}[0]\tilde{h}[0] + \tilde{x}[1]\tilde{h}[4] + \tilde{x}[2]\tilde{h}[3] + \tilde{x}[3]\tilde{h}[2] + \tilde{x}[4]\tilde{h}[1] = -13,$

$\tilde{y}[1] = \sum_{r=0}^{4} \tilde{x}[r]\tilde{h}[1-r] = \tilde{x}[0]\tilde{h}[1] + \tilde{x}[1]\tilde{h}[0] + \tilde{x}[2]\tilde{h}[4] + \tilde{x}[3]\tilde{h}[3] + \tilde{x}[4]\tilde{h}[2] = -13,$

$\tilde{y}[2] = \sum_{r=0}^{4} \tilde{x}[r]\tilde{h}[2-r] = \tilde{x}[0]\tilde{h}[2] + \tilde{x}[1]\tilde{h}[1] + \tilde{x}[2]\tilde{h}[0] + \tilde{x}[3]\tilde{h}[4] + \tilde{x}[4]\tilde{h}[3] = -13,$

$\tilde{y}[3] = \sum_{r=0}^{4} \tilde{x}[r]\tilde{h}[3-r] = \tilde{x}[0]\tilde{h}[3] + \tilde{x}[1]\tilde{h}[2] + \tilde{x}[2]\tilde{h}[1] + \tilde{x}[3]\tilde{h}[0] + \tilde{x}[4]\tilde{h}[4] = -13,$

$\tilde{y}[4] = \sum_{r=0}^{4} \tilde{x}[r]\tilde{h}[4-r] = \tilde{x}[0]\tilde{h}[4] + \tilde{x}[1]\tilde{h}[3] + \tilde{x}[2]\tilde{h}[2] + \tilde{x}[3]\tilde{h}[1] + \tilde{x}[4]\tilde{h}[0] = -13.$

Therefore, $\tilde{y}[n] = \{-13, \quad -13, \quad -13, \quad -13, \quad -13,\}, 0 \le n \le 4.$

(b) $\tilde{y}[n] = \{1, \quad 1, \quad 1, \quad 1, \quad 1,\}, 0 \le n \le 4.$

5.3 Since $\tilde{\psi}_k[n+rN] = \tilde{\psi}_k[n]$, hence all the terms which are not in the range $0, 1, \ldots, N-1$, can be accumulated to $\tilde{\psi}_k[n]$, where $0 \le k \le N-1$. Hence, in this case the Fourier series representation involves only N complex exponential sequences. Let

$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j2\pi kn/N}$, then

$$\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi rn/N} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j2\pi(k-r)n/N} = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \left(\sum_{n=0}^{N-1} e^{j2\pi(k-r)n/N} \right).$$

Now, from Eq. (5.11), the inner summation is equal to N if $k = r$, otherwise it is equal to 0. Thus, $\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi rn/N} = \tilde{X}[r]$. Next, we observe $\tilde{X}[k+\ell N]$

$$= \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi(k+\ell N)n/N} = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi kn/N} e^{-j2\pi \ell n} = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi kn/N} = \tilde{X}[k].$$

5.4 **(a)** $\tilde{x}_1[n]=\cos\left(\frac{\pi n}{4}\right)=\frac{1}{2}\left\{e^{j\pi n/4}+e^{-j\pi n/4}\right\}$ The period of $\tilde{x}_1[n]$ is $N=8$.

$$\tilde{X}_1[k]=\frac{1}{2}\left\{\sum_{n=0}^{7}e^{j2\pi n/8}e^{-j2\pi kn/8}+\sum_{n=0}^{7}e^{-j2\pi n/8}e^{-j2\pi kn/8}\right\}$$

$$=\frac{1}{2}\left\{\sum_{n=0}^{7}e^{-j2\pi n(k-1)/8}+\sum_{n=0}^{7}e^{-j2\pi n(k+1)/8}\right\}.$$ Now, from Eqn. (5.11) we observe

$$\sum_{n=0}^{7}e^{-j2\pi n(k-1)/8}=\begin{cases}8, & \text{for } k=1,\\ 0, & \text{otherwise,}\end{cases} \text{ and } \sum_{n=0}^{7}e^{-j2\pi n(k+1)/8}=\begin{cases}8, & \text{for } k=7,\\ 0, & \text{otherwise.}\end{cases}$$

Hence, $\tilde{X}_1[k]=\begin{cases}4, & \text{for } k=1,7,\\ 0, & \text{otherwise.}\end{cases}$

(b) $\tilde{x}_2[n]=\sin\left(\frac{\pi n}{3}\right)+3\cos\left(\frac{\pi n}{4}\right)=\frac{1}{2j}\{e^{j\pi n/3}-e^{-j\pi n/3}\}+\frac{3}{2}\{e^{j\pi n/4}+e^{-j\pi n/4}\}$. The period of $\sin\left(\frac{\pi n}{3}\right)$ is 6 and the period of $\cos\left(\frac{\pi n}{4}\right)$ is 8. Hence, the period of $\tilde{x}_2[n]$ is the GCM of $(6,8)$ and is 24.

$$\tilde{X}_2[k]=\frac{1}{2j}\left\{\sum_{n=0}^{23}e^{j8\pi n/24}e^{-j2\pi kn/24}-\sum_{n=0}^{23}e^{-j8\pi n/24}e^{-j2\pi kn/24}\right\}$$

$$+\frac{3}{2}\left\{\sum_{n=0}^{23}e^{j6\pi n/24}e^{-j2\pi kn/24}-\sum_{n=0}^{23}e^{-j6\pi n/24}e^{-j2\pi kn/24}\right\}$$

$$=\frac{1}{2j}\left\{\sum_{n=0}^{23}e^{-j2\pi n(k-3)/24}-\sum_{n=0}^{23}e^{-j2\pi n(k+3)/24}\right\}$$

$$+\frac{3}{2}\left\{\sum_{n=0}^{23}e^{-j2\pi n(k-4)/24}-\sum_{n=0}^{23}e^{-j2\pi n(k+4)/24}\right\}.$$ Hence $\tilde{X}_2[k]=\begin{cases}-j12, & k=3,\\ j12, & k=21,\\ 36, & k=4,20,\\ 0, & \text{otherwise.}\end{cases}$

5.5 Let $\tilde{P}[k]$ denote the coefficients of the Fourier series representation of $\tilde{p}[n]$. Since $\tilde{p}[n]$ is periodic with a period N, then from Eq. (5.185b), we have

$\tilde{P}[k]=\sum_{n=0}^{N-1}\tilde{p}[n]e^{-j2\pi kn/N}=1$. Hence, from Eq. (5.185a) we get

$$\tilde{p}[n]=\frac{1}{N}\sum_{\ell=0}^{N-1}\tilde{P}[\ell]e^{j2\pi\ell n/N}=\frac{1}{N}\sum_{\ell=0}^{N-1}e^{j2\pi\ell n/N}.$$

5.6 $\tilde{X}[k]=X(e^{j\omega})\Big|_{\omega=2\pi k/N}=X(e^{j2\pi k/N})=\sum_{n=-\infty}^{\infty}x[n]e^{-j2\pi k/N}$, $-\infty<k<\infty$.

Now, $\tilde{X}[k+\ell N]=X(e^{j2\pi(k+\ell N)/N})=X(e^{j2\pi k/N}e^{j\pi\ell})=X(e^{j2\pi k/N})=\tilde{X}[k]$.

Likewise, $\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j2\pi kn/N} = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{\ell=-\infty}^{\infty}x[\ell]e^{-j2\pi\ell n/N}\right)e^{j2\pi kn/N}$

$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{\ell=-\infty}^{\infty}x[\ell]e^{j2\pi(k-\ell)n/N}$. Let $\ell = n + rN$. Then

$\tilde{x}[n] = \frac{1}{N}\sum_{r=-\infty}^{\infty}x[n+rN]\left(\sum_{k=0}^{N-1}e^{-j2\pi kr}\right)$. But $\sum_{k=0}^{N-1}e^{-j2\pi kr} = N$. Hence,

$\tilde{x}[n] = \sum_{r=-\infty}^{\infty}x[n+rN].$

5.7 **(a)** $\tilde{G}[k] = \sum_{n=0}^{N-1}\tilde{g}[n]e^{-j2\pi kn/N} = \sum_{n=0}^{N-1}\tilde{x}[n]\tilde{y}[n]\,e^{-j2\pi kn/N}$. Now,

$\tilde{x}[n] = \frac{1}{N}\sum_{r=0}^{N-1}\tilde{X}[r]e^{-j2\pi rn/N}$. Therefore,

$\tilde{G}[k] = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{r=0}^{N-1}\tilde{X}[r]\,\tilde{y}[n]e^{-j2\pi(k-r)n/N} = \frac{1}{N}\sum_{r=0}^{N-1}\tilde{X}[r]\sum_{n=0}^{N-1}\tilde{y}[n]e^{-j2\pi(k-r)n/N}$

$= \frac{1}{N}\sum_{r=0}^{N-1}\tilde{X}[r]\tilde{Y}[k-r].$

(b) $\tilde{h}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\tilde{Y}[k]\,e^{j2\pi kn/N} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{r=0}^{N-1}\tilde{x}[r]\tilde{Y}[k]\,e^{j2\pi k(n-r)/N}$

$= \sum_{r=0}^{N-1}\tilde{x}[r]\left(\frac{1}{N}\sum_{k=0}^{N-1}\tilde{Y}[k]\,e^{j2\pi k(n-r)/N}\right) = \sum_{r=0}^{N-1}\tilde{x}[r]\,\tilde{y}[n-r].$

5.8 **(a)** $x_a[n] = \sin(2\pi n/N) = \frac{1}{2j}(e^{j2\pi n/N} - e^{-j2\pi n/N})$. Therefore,

$X_a[k] = \frac{1}{2j}\sum_{n=0}^{N-1}e^{j2\pi n/N}e^{-j2\pi kn/N} - \frac{1}{2j}\sum_{n=0}^{N-1}e^{-j2\pi n/N}e^{-j2\pi kn/N}$

$= \frac{1}{2j}\sum_{n=0}^{N-1}e^{-j2\pi(k-1)n/N} - \frac{1}{2j}\sum_{n=0}^{N-1}e^{-j2\pi(k+1)n/N}$. From Eq. (5.11), the first sum is equal to N when $k=1$ and 0 othereise. Likewise, from Eq. (5.11), the second sum is equal to N when $k = N-1$ and 0 otherwise. Therefore,

$$X_a[k] = \begin{cases} N/2j, & k=1, \\ -N/2j, & k = N-1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $x_b[n] = \cos^2\left(\frac{2\pi n}{N}\right) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{4\pi n}{N}\right)$. Now the N–point DFT of $\frac{1}{2}$ is $\frac{N}{2}$ for $k=0$ and 0 otherwise. From Example 5.2, the N–point DFT of $\cos\left(\frac{4\pi n}{N}\right)$ is $\frac{N}{2}$ for $k=2$ and $k = N-2$ and 0 otherwise. Therefore,

$$X_b[k]=\begin{cases} N/2, & k=0, \\ N/4, & k=2, N-2, \\ 0, & \text{otherwise.} \end{cases}$$

(c) $x_c[n]=\cos^3\left(\frac{2\pi n}{N}\right)=\frac{1}{4}\cos\left(\frac{6\pi n}{N}\right)+\frac{3}{4}\cos\left(\frac{2\pi n}{N}\right)$. From Example 5.2, the N–point DFT of $\cos\left(\frac{6\pi n}{N}\right)$ is $\frac{N}{2}$ for $k=3$ and $k=N-3$ and 0 otherwise. Likewise, from Example 5.2, the N–point DFT of $\cos\left(\frac{2\pi n}{N}\right)$ is $\frac{N}{2}$ for $k=1$ and $k=N-1$ and 0 otherwise. Therefore, $X_c[k]=\begin{cases} N/8, & k=3, N-3, \\ 3N/8, & k=1, N-1, \\ 0, & \text{otherwise.} \end{cases}$

5.9 **(a)** $Y_a[k]=\sum_{n=0}^{N-1}\alpha^n W_N^{kn}=\sum_{n=0}^{N-1}(\alpha W_N^k)^n=\frac{1-\alpha W_N^{kN}}{1-\alpha W_N^k}=\frac{1-\alpha}{1-\alpha W_N^k}$.

(b) $Y_b[k]=2\sum_{k\ \text{even}}W_N^{kn}-3\sum_{k\ \text{odd}}W_N^{kn}$. Assume first N is even, i.e., $N=2L$. Then

$$Y_b[k]=2\sum_{r=0}^{L-1}W_{2L}^{k2r}-3\sum_{r=0}^{L-1}W_{2L}^{k(2r+1)}=2\sum_{r=0}^{L-1}W_L^{kr}-3W_{2L}^k\sum_{r=0}^{L-1}W_L^{kr}=\left(2-3W_N^k\right)\left(\frac{1-W_L^{kL}}{1-W_L^k}\right)=0.$$

Next, assume N is odd, i.e., $N=2L+1$. Then $Y_b[k]=2\sum_{r=0}^{L}W_{2L}^{k2r}-3\sum_{r=0}^{L-1}W_{2L}^{k(2r+1)}$

$$=2\sum_{r=0}^{L}W_L^{kr}-3W_{2L}^k\sum_{r=0}^{L-1}W_L^{kr}=2\left(\frac{1-W_L^{k(L+1)}}{1-W_L^k}\right)-3W_{2L}^k\left(\frac{1-W_L^{kL}}{1-W_L^k}\right)=2\left(\frac{1-W_L^k}{1-W_L^k}\right)=2.$$

5.10 $x[n]=\cos(\omega_o n)=\frac{1}{2}(e^{j\omega_o n}+e^{-j\omega_o n}),\ 0\le n\le N-1$. Therefore,

$$X[k]=\frac{1}{2}\sum_{n=0}^{N-1}e^{j\omega_o n}e^{-j2\pi kn/N}+\frac{1}{2}\sum_{n=0}^{N-1}e-j^{j\omega_o n}e^{-j2\pi kn/N}$$

$$=\frac{1}{2}\sum_{n=0}^{N-1}e^{-j(\frac{2\pi k}{N}-\omega_o)n}+\frac{1}{2}\sum_{n=0}^{N-1}e^{-j(\frac{2\pi k}{N}+\omega_o)n}$$

$$=\frac{1}{2}\cdot\frac{1-e^{-j\left(\frac{2\pi k}{N}-\omega_o\right)N}}{1-e^{-j\left(\frac{2\pi k}{N}-\omega_o\right)}}+\frac{1}{2}\cdot\frac{1-e^{-j\left(\frac{2\pi k}{N}+\omega_o\right)N}}{1-e^{-j\left(\frac{2\pi k}{N}+\omega_o\right)}}$$

$$= \frac{1}{2} e^{-j\left(\frac{2\pi k}{N}-\omega_o\right)\left(\frac{N-1}{2}\right)} \cdot \frac{\sin\left(\pi k - \frac{\omega_o N}{2}\right)}{\sin\left(\frac{\pi k}{N} - \frac{\omega_o}{2}\right)} + \frac{1}{2} e^{-j\left(\frac{2\pi k}{N}+\omega_o\right)\left(\frac{N-1}{2}\right)} \cdot \frac{\sin\left(\pi k + \frac{\omega_o N}{2}\right)}{\sin\left(\frac{\pi k}{N} + \frac{\omega_o}{2}\right)}.$$

5.11 $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} = \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{(2r+1)k}$

$= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_{N/2}^{rk}$

$= \sum_{r=0}^{(N/2)-1} x_0[r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x_1[r] W_{N/2}^{rk}$

$= X_0[\langle k \rangle_{N/2}] + W_N^k X_1[\langle k \rangle_{N/2}],\ 0 \le k \le N-1.$

5.12 $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} = \sum_{n=0}^{(N/2)-1} x[n] W_N^{nk} + \sum_{n=N/2}^{N-1} x[n] W_N^{nk}$

$= \sum_{n=0}^{(N/2)-1} x[n] W_N^{nk} + W_N^{(N/2)k} \sum_{n=0}^{(N/2)-1} x[\frac{N}{2}+n] W_N^{nk}$

$= \sum_{n=0}^{(N/2)-1} \left(x[n] + (-1)^k x[\frac{N}{2}+n] \right) W_N^{nk}$. For $k = 2\ell$, we get

$X[2\ell] = \sum_{n=0}^{(N/2)-1} \left(x[n] + x[\frac{N}{2}+n] \right) W_N^{2n\ell} = \sum_{n=0}^{(N/2)-1} \left(x[n] + x[\frac{N}{2}+n] \right) W_{N/2}^{n\ell} = X_0[\ell]$

and for $k = 2\ell+1$ we get $X[2\ell+1] = \sum_{n=0}^{(N/2)-1} \left(x[n] - x[\frac{N}{2}+n] \right) W_N^{(2\ell+1)n}$

$= \sum_{n=0}^{(N/2)-1} \left(x[n] + x[\frac{N}{2}+n] \right) W_N^n \cdot W_{N/2}^{n\ell} = X_1[\ell]$ where $0 \le \ell \le \frac{N}{2} - 1.$

5.13 $g[n] = \frac{1}{2}(x[2n] + x[2n+1]),\ h[n] = \frac{1}{2}(x[2n] - x[2n+1]),\ 0 \le n \le \frac{N}{2} - 1.$ Solving for $x[2n]$ and $x[2n+1]$, we get $x[2n] = g[n] + h[n]$ and $x[2n+1] = g[n] - h[n]$. Therefore,

$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} = \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{(2r+1)k}$

$= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_{N/2}^{rk}$

$= \sum_{r=0}^{(N/2)-1} (g[n] + h[n]) W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} (g[n] - h[n]) W_{N/2}^{rk}$

$= (1 + W_N^k) \sum_{r=0}^{(N/2)-1} g[n] W_{N/2}^{rk} + (1 - W_N^k) \sum_{r=0}^{(N/2)-1} h[n] W_{N/2}^{rk}$

$= (1+W_N^k)G[\langle k\rangle_{N/2}] + (1-W_N^k)H[\langle k\rangle_{N/2}],\ 0 \le k \le N-1.$

5.14 $g[n] = a_1 x[2n] + a_2 x[2n+1],\ h[n] = a_3 x[2n] - a_4 x[2n+1],\ 0 \le n \le \frac{N}{2}-1,$ with $a_1 a_4 \ne a_2 a_3$. Solving for $x[2n]$ and $x[2n+1]$, we get $x[2n] = \dfrac{a_4 g[n] - a_2 h[n]}{a_1 a_4 - a_2 a_3}$ and $x[2n+1] = \dfrac{-a_3 g[n] + a_1 h[n]}{a_1 a_4 - a_2 a_3}$. Therefore,

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} = \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{(2r+1)k}$$

$$= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_{N/2}^{rk}$$

$$= \sum_{r=0}^{(N/2)-1} \left(\frac{a_4 g[r] - a_2 h[r]}{a_1 a_4 - a_2 a_3} \right) W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} \left(\frac{-a_3 g[n] + a_1 h[n]}{a_1 a_4 - a_2 a_3} \right) W_{N/2}^{rk}$$

$$= \frac{1}{a_1 a_4 - a_2 a_3} \Big((a_4 - a_3 W_N^k) G[\langle k\rangle_{N/2}] + (-a_2 + a_1 W_N^k) H[\langle k\rangle_{N/2}] \Big),\ 0 \le n \le N-1.$$

5.15 **(a)** $G[k] = \sum_{n=0}^{N-1} x[n] W_{2N}^{nk}$. For k even, i.e., $k = 2\ell$, $G[2\ell] = \sum_{n=0}^{N-1} x[n] W_{2N}^{2\ell n}$

$$= \sum_{n=0}^{N-1} x[n]_{2N}^{\ell n} = X[\ell],\ 0 \le \ell \le N-1.$$

(b) $H[k] = \sum_{n=N}^{2N-1} x[n-N] W_{2N}^{nk}$. Let $m = n - N$ or $n = m + N$. Then

$H[k] = \sum_{m=0}^{N-1} x[m] W_{2N}^{(m+N)k} = (-1)^k \sum_{m=0}^{N-1} x[m] W_{2N}^{mk}$. For k even, i.e., $k = 2\ell$,

$$H[2\ell] = \sum_{n=0}^{N-1} x[n] W_{2N}^{2\ell n} = \sum_{n=0}^{N-1} x[n] W_N^{\ell n} = X[\ell],\ 0 \le \ell \le N-1.$$

5.16 $Y[k] = \sum_{n=0}^{2N-1} y[n] W_{2N}^{nk} = \sum_{n=0}^{2N-1} (g[n] + h[n]) W_{2N}^{nk} = G[k] + H[k]$. For k even, i.e., $k = 2\ell$, $Y[2\ell] = G[2\ell] + H[2\ell] = 2X[\ell],\ 0 \le \ell \le N-1$.

For k odd, i.e., $k = 2\ell+1$, $G[2\ell+1] = \sum_{n=0}^{N-1} x[n] W_{2N}^{(2\ell+1)n} = \sum_{n=0}^{N-1} x[n] W_{2N}^{n} W_N^{\ell n}$ and

$$H[2\ell+1] = -\sum_{n=0}^{N-1} x[n] W_{2N}^{(2\ell+1)n} = -\sum_{n=0}^{N-1} x[n] W_{2N}^{n} W_N^{\ell n} = -G[2\ell+1],\ 0 \le \ell \le N-1.$$

Hence, for $k = 2\ell+1$, $Y[2\ell+1] = G[2\ell+1] - G[2\ell+1] = 0,\ 0 \le \ell \le N-1$.

5.17 $Y[k] = \sum_{n=0}^{MN-1} y[n]W_{MN}^{nk} = \sum_{n=0}^{N-1} x[n]W_{MN}^{nk}$. Thus,

$Y[kM] = \sum_{n=0}^{N-1} x[n]W_{MN}^{nkM} = \sum_{n=0}^{N-1} x[n]W_{N}^{nk} = X[k].$

5.18

5.19 **(a)** Now, $X[N/2] = \sum_{n=0}^{N-1} x[n]W_N^{nN/2} = \sum_{n=0}^{N-1} (-1)^n x[n]$. Hence if $x[n] = x[N-1-n]$ and N is even, then $X[N/2] = \sum_{n=0}^{N-1} (-1)^n x[n] = 0.$

(b) $X[0] = \sum_{n=0}^{N-1} x[n]$. Hence if $x[n] = -x[N-1-n]$, then $X[0] = 0.$

(c) $X[2\ell] = \sum_{n=0}^{N-1} x[n]W_N^{2n\ell} = \sum_{n=0}^{(N/2)-1} x[n]W_N^{2n\ell} + \sum_{n=(N/2)-1}^{N-1} x[n]W_N^{2n\ell}$

$= \sum_{n=0}^{(N/2)-1} x[n]W_N^{2n\ell} + \sum_{n=0}^{(N/2)-1} x[n+\frac{N}{2}]W_N^{2n\ell} = \sum_{n=0}^{M-1} \left(x[n]+x[n+M]\right)W_{2M}^{2n\ell}$. Hence if $x[n] = -x[n+M]$, then $X[2\ell] = 0$ for $0 \le \ell \le M-1$.

5.20 $X[2m] = \sum_{n=0}^{N-1} x[n]W_N^{2mn} = \sum_{n=0}^{(N/2)-1} x[n]W_N^{2mn} + \sum_{n=N/2}^{N-1} x[n]W_N^{2mn}$

$= \sum_{n=0}^{(N/2)-1} x[n]W_N^{2mn} + \sum_{n=0}^{N/2-1} x[n+\frac{N}{2}]W_N^{2mn}W_N^{mN} = \sum_{n=0}^{(N/2)-1} \left(x[n]+x[n+\frac{N}{2}\right)W_N^{2mn} = 0,$

$0 \le m \le \frac{N}{2}-1$. This implies $x[n]+x[n+\frac{N}{2}] = 0.$

5.21 **(a)** Using the circular convolution property of the DFT given in Table 5.3, we get $\text{DFT}\{x[\langle n-m_1\rangle_N\} = W_N^{km_1}X[k]$ and $\text{DFT}\{x[\langle n-m_2\rangle_N\} = W_N^{km_2}X[k]$.
Hence, $W[k] = \text{DFT}\{w[n]\} = W_N^{km_1}X[k] + W_N^{km_2}X[k] = (W_N^{km_1} + W_N^{km_1})X[k].$

(b) $g[n] = \frac{1}{2}\left(x[n] + (-1)^n x[n]\right) = \frac{1}{2}\left(x[n] + W_N^{-(N/2)n}x[n]\right)$ Using the circular convolution property of the DFT given in Table 5.3, we get
$G[k] = \text{DFT}\{g[n]\} = \frac{1}{2}\left\{X[k] + X[\langle k-\frac{N}{2}\rangle_N\right\}$

(c) Using the circular convolution property of the DFT given in Table 5.3, we get
$Y[k] = \text{DFT}\{y[n]\} = X[k]\cdot X[k] = X^2[k].$

5.22 **(a)** $\text{DFT}\left\{x[\langle n-\frac{N}{2}\rangle_N]\right\} = W_N^{k(N/2)}X[k] = (-1)^k X[k].$ Hence,

$$U[k] = \text{DFT}\{u[n]\} = \text{DFT}\left\{x[n] - x[\langle n-\frac{N}{2}\rangle_N]\right\} = X[k] - (-1)^k X[k] = \begin{cases} 2X[k], & \text{for } k \text{ odd,} \\ 0, & \text{for } k \text{ even.} \end{cases}$$

(b) $V[k] = \text{DFT}\{v[n]\} = \text{DFT}\left\{x[n] - x[n-\frac{\text{N}}{2}]\right\} = X[k] + X[k] = 2X[k].$

(c) $y[n] = (-1)^n x[n] = W_N^{(N/2)n} x[n].$ Hence, using the circular frequency-shifting property of the DFT given in Table 5.3, we get

$Y[k] = \text{DFT}\{y[n]\} = \text{DFT}\left\{W_N^{(N/2)n} x[n]\right\} = X[\langle k-\frac{N}{2}\rangle_N].$

5.23 **(a)** From the circular frequency-shifting property of the DFT given in Table 5.3, we get

$\text{IDFT}\{X[\langle k-m_1\rangle_N\} = W_N^{-m_1 n} x[n]$ and $\text{IDFT}\{X[\langle k-m_2\rangle_N\} = W_N^{-m_2 n} x[n].$ Hence,

$w[n] = \text{IDFT}\{W[k]\} = \text{IDFT}\{\alpha X[\langle k-m_1\rangle_N\} + \beta X[\langle k-m_2\rangle_N\}$

$= \alpha W_N^{-m_1 n} x[n] + \beta W_N^{-m_2 n} x[n] = (\alpha W_N^{-m_1 n} + \beta W_N^{-m_2 n}) x[n].$

(b) $G[k] = \frac{1}{2}(X[k] + (-1)^k X[k]) = \frac{1}{2}\left(X[k] + W_N^{-(N/2)k} X[k]\right).$ Using the circular time-shifting property of the DFT given in Table 5.3, we get

$g[n] = \text{IDFT}\{G[k]\} = \frac{1}{2}\left(x[n] + x[\langle n-\frac{N}{2}\rangle_N\right).$

(c) Using the modulation property of the DFT given in Table 5.3, we get

$y[n] = \text{IDFT}\{Y[k]\} = N \cdot x[n] \cdot x[n] = N \cdot x^2[n].$

5.24 **(a)** $X[2m] = \sum_{n=0}^{N-1} x[n] W_N^{2mn} = \sum_{n=0}^{(N/2)-1} x[n] W_N^{2mn} + \sum_{n=N/2}^{N-1} x[n] W_N^{2mn}$

$$= \sum_{n=0}^{(N/2)-1} x[n] W_N^{2mn} + \sum_{n=0}^{N/2-1} x[n+\frac{N}{2}] W_N^{2m(n+\frac{N}{2})}$$

$$= \sum_{n=0}^{(N/2)-1} x[n] W_N^{2mn} + \sum_{n=0}^{N/2-1} x[n+\frac{N}{2}] W_N^{2mn} W_N^{mn}$$

$$= \sum_{n=0}^{(N/2)-1} \left(x[n] + x[n+\frac{N}{2}]\right) W_N^{2mn} = \sum_{n=0}^{(N/2)-1} \left(x[n] - x[n]\right) W_N^{2mn} = 0,\ 0 \le m \le \frac{N}{2} - 1.$$

(b) $X[4\ell] = \sum_{n=0}^{N-1} x[n] W_N^{4\ell n}$

$$= \sum_{n=0}^{(N/4)-1} x[n] W_N^{4\ell n} + \sum_{n=N/4}^{(N/2)-1} x[n] W_N^{4\ell n} + \sum_{n=N/2}^{(3N/4)-1} x[n] W_N^{4\ell n} + \sum_{n=3N/4}^{N-1} x[n] W_N^{4\ell n}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left(x[n]W_N^{4\ell n} + x[n+\frac{N}{4}]W_N^{4\ell(n+\frac{N}{4})} + x[n+\frac{N}{2}]W_N^{4\ell(n+\frac{N}{2})} + x[n+\frac{3N}{4}]W_N^{4\ell(n+\frac{3N}{4})} \right)$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left(x[n] + x[n+\frac{N}{4}]W_N^{\ell n} + x[n+\frac{N}{2}]W_N^{2\ell n} + x[n+\frac{3N}{4}]W_N^{3\ell n} \right) W_N^{4\ell n}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} (x[n] - X[n] + x[n] - x[n]) W_N^{4\ell n} = 0 \text{ as } W_N^{\ell N} = W_N^{2\ell N} = W_N^{3\ell N} = 1.$$

5.25 **(a)** $X[N-k] = \sum_{n=0}^{N-1} x[n] W_N^{(N-k)n} = \sum_{n=0}^{N-1} x[n] W_N^{-kn} + X^*[k].$

(b) $X[0] = \sum_{n=0}^{N-1} x[n] W_N^0 \sum_{n=0}^{N-1} x[n]$ which is real.

(c) $X[\frac{N}{2}] = \sum_{n=0}^{N-1} x[n] W_N^{(N/2)n} = \sum_{n=0}^{N-1} (-1)^n x[n]$ which is real.

5.26 $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}.$

(a) $X^*[k] = \sum_{n=0}^{N-1} x^*[n] W_N^{-nk}$. Replacing n by $N-n$ in the summation we obtain

$X^*[k] = \sum_{n=0}^{N-1} x^*[N-n] W_N^{-(N-n)k} = \sum_{n=0}^{N-1} x^*[N-n] W_N^{nk}$. Thus,

$\text{DFT}\{x^*[N-n]\} = \text{DFT}\{x^*[\langle -n\rangle_N\} = X^*[k].$

(b) $\text{Re}\{x[n]\} = \frac{1}{2}\{x[n] + x^*[n]\}$. Taking the DFT of both sides and using the results of Part (a) we get $\text{DFT}\{\text{Re}\{x[n]\}\} = \frac{1}{2}\{X[k] + X^*[\langle -k\rangle_N\}.$

(c) $j\,\text{Im}\{x[n]\} = \frac{1}{2}\{x[n] - x^*[n]\}$. Thus, $\text{DFT}\{j\,\text{Im}\{x[n]\}\} = \frac{1}{2}\{X[k] - X^*[\langle -k\rangle_N\}.$

(d) $x_{cs}[n] = \frac{1}{2}\{x[n] + x^*[\langle -n\rangle_N]\}$. Using the linearity property and results of Part (b) we get $\text{DFT}\{x_{cs}[n]\} = \frac{1}{2}\{X[k] + X^*[k]\} = \text{Re}\{X[k]\}.$

(e) $x_{ca}[n] = \frac{1}{2}\{x[n] - x^*[\langle -n\rangle_N]\}$. Using the linearity property and results of Part (b) we get $\text{DFT}\{x_{ca}[n]\} = \frac{1}{2}\{X[k] - X^*[k]\} = j\,\text{Im}\{X[k]\}.$

5.27 Since for a real sequence, $x[n] = x^*[n]$, taking the DFT of both sides we get $X[k] = X^*[\langle -k\rangle_N]$. This implies $\text{Re}\{X[k]\} + j\,\text{Im}\{X[k]\} = \text{Re}\{X[\langle -k\rangle_N]\} - j\,\text{Im}\{X[\langle -k\rangle_N]\}$.
Comparing real and imaginary parts we get $\text{Re}\{X[k]\} = \text{Re}\{X[\langle -k\rangle_N]\}$ and $\text{Im}\{X[k]\} = -\text{Im}\{X[\langle -k\rangle_N]\}$.

Also, $|X[k]| = \sqrt{(\text{Re}\{X[k]\})^2 + (\text{Im}\{X[k]\})^2}$
$= \sqrt{(\text{Re}\{X[\langle -k\rangle_N]\})^2 + (\text{Im}\{X[\langle -k\rangle_N]\})^2} = |X[\langle -k\rangle_N]|$ and

$$\arg\{X[k]\} = \tan^{-1}\left(\frac{\text{Im}\{X[k]\}}{\text{Re}\{X[k]\}}\right) = \tan^{-1}\left(\frac{-\text{Im}\{X[\langle -k\rangle_N]\}}{\text{Re}\{X[\langle -k\rangle_N]\}}\right) = -\arg\{X[\langle -k\rangle_N]\}.$$

5.28 **(a)** $x_1[\langle -n\rangle_9] = \{4 \quad 3 \quad -5 \quad 1 \quad -2 \quad -2 \quad 1 \quad -5 \quad 3\} = x_1[n]$. Thus, $x_1[n]$ is a circular even sequence and hence, it has a real-valued 9-point DFT.

(b) $x_2[\langle -n\rangle_9] = \{0 \quad -5 \quad -1 \quad -4 \quad 3 \quad -3 \quad 4 \quad 1 \quad 5\} = -x_2[n]$. Thus, $x_2[n]$ is a circular odd sequence and hence, it has an imaginaryl-valued 9-point DFT.

(c) $x_3[\langle -n\rangle_9] = \{0 \quad -5 \quad -1 \quad -4 \quad 3 \quad -3 \quad 4 \quad 2 \quad -5\}$ which is neither equal to $x_3[n]$ nor equal to $-x_3[n]$. Thus, $x_3[n]$ has a complex-valued 9-point DFT.

(d) $x_4[\langle -n\rangle_9] = \{-5 \quad 5 \quad -2 \quad 2 \quad 4 \quad 4 \quad 2 \quad -2 \quad 5\} = x_4[n]$. Thus, $x_4[n]$ is a circular even sequence and hence, it has a real-valued 9-point DFT.

5.29 **(a)** $h[n] = g[\langle n-5\rangle_8]$. Hence, $H[k] = W_8^{-5k}G[k] = e^{j10\pi k/8}G[k] = e^{j5\pi k/4}G[k]$
$= \{2.6 + j4.1,\ e^{j5\pi/4}(3 - j2.7),\ e^{j5\pi/2}(-4.2 + j1.4),\ e^{j15\pi/4}(3.5 - j2.6),$
$e^{j5\pi}(0.5),\ e^{j25\pi/4}(1.3 + j4.4),\ e^{j15\pi/2}(2.4 - j1.6),\ e^{j35\pi/4}(-3 + j1.6)\}$.

(b) $H[k] = G[\langle k+3\rangle_8]$. Hence, $h[n] = W_8^3 g[n] = e^{-j6\pi n/8}g[n] = e^{-j3\pi n/4}g[n]$
$= \{-0.1 - j0.7,\ e^{-j3\pi/4}(1.3 + j),\ e^{-j3\pi/2}(2 + j0.7),\ e^{-j9\pi/4}(1.1 + j2.2),$
$e^{-j3\pi}(-0.8 + j0.2),\ e^{-j15\pi/4}(3.4 - j0.1),\ e^{-j9\pi/2}(-1.2 + j3.1),\ e^{-j21\pi/4}(j1.5)\}$.

5.30 **(a)** $y[n] = \alpha g[n] + \beta h[n]$. Therefore,

$$Y[k] = \sum_{n=0}^{N-1} y[n]W_N^{nk} = \alpha\sum_{n=0}^{N-1} g[n]W_N^{nk} + \beta\sum_{n=0}^{N-1} h[n]W_N^{nk} = \alpha G[k] + \beta H[k].$$

(b) $x[n] = g[\langle n - n_o\rangle_N]$. Therefore, $X[k] = \sum_{n=0}^{N-1} g[\langle n-n_o\rangle_N]W_N^{nk}$

$$= \sum_{n=0}^{n_o-1} g[N+n-n_o]W_N^{nk} + \sum_{n=n_o}^{N-1} g[n-n_o]W_N^{nk}$$

$$= \sum_{n=N-n_o}^{N-1} g[n]W_N^{(n+n_o-N)k} + \sum_{n=0}^{N-n-1} g[n_o]W_N^{(n+n_o)k} = W_N^{n_o k}\sum_{n=0}^{N-1} g[n]W_N^{nk} = W_N^{n_o k}G[k].$$

(c) $u[n] = W_N^{-k_o n} g[n]$. Therefore, $U[k] = \sum_{n=0}^{N-1} u[n]W_N^{kn} = \sum_{n=0}^{N-1} g[n]W_N^{(k-k_o)n}$

$$= \begin{cases} \sum_{n=0}^{N-1} g[n]W_N^{(k-k_o)n}, & \text{if } k \geq k_o, \\ \sum_{n=0}^{N-1} g[n]W_N^{(N+k-k_o)n}, & \text{if } k < k_o. \end{cases}$$

Thus, $U[k] = \begin{cases} G[k-k_o], & \text{if } k \geq k_o, \\ G[N+k-k_o], & \text{if } k < k_o, \end{cases} = G[\langle k-k_o\rangle_N]$.

(d) $h[n] = G[n]$. Therefore, $H[k] = \sum_{n=0}^{N-1} h[n]W_N^{nk} = \sum_{n=0}^{N-1} G[n]W_N^{nk}$

$= \sum_{n=0}^{N-1}\sum_{r=0}^{N-1} g[r]W_N^{nr}W_N^{kr} = \sum_{r=0}^{N-1} g[r]\sum_{n=0}^{N-1} W_N^{(k+r)n}$. The second sum is nonzero only if $k = r = 0$ or else if $r = N-k$ and $k \neq 0$. Hence,

$$H[k] = \begin{cases} Ng[0], & \text{if } k = 0, \\ Ng[N-k], & \text{if } k > 0, \end{cases} = Ng[\langle -k\rangle_N].$$

(e) $u[n] = \sum_{m=0}^{N-1} g[m]h[\langle -n-m\rangle_N]$. Therefore, $U[k] = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} g[m]h[\langle -n-m\rangle_N]W_N^{nk}$

$$= \sum_{m=0}^{N-1} g[m]\sum_{n=0}^{N-1} h[\langle -n-m\rangle_N]W_N^{nk} = \sum_{m=0}^{N-1} g[m]H[k]W_N^{mk} = H[k]G[k].$$

5.31 $\sum_{n=0}^{N-1} g[n]h^*[n] = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} G[k]W_N^{-nk}h^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} G[k]\sum_{n=0}^{N-1} h^*[n]W_N^{-nk}$

$$= \frac{1}{N}\sum_{k=0}^{N-1} G[k]H^*[k].$$

5.32 $\text{DFT}\{r_{xy}[\ell]\} = \mathcal{R}_{xy}[\ell] = \sum_{\ell=0}^{N-1} r_{xy}[\ell]W_N^{\ell k} = \sum_{\ell=0}^{N-1}\left(\sum_{n=0}^{N-1} x[n]y[\langle \ell+n\rangle_N\right)W_N^{\ell k}$

$$= \sum_{n=0}^{N-1} x[n]\left(\sum_{\ell=0}^{N-1} y[\langle \ell+n\rangle_N]W_N^{\ell k}\right) = \sum_{n=0}^{N-1} x[n]\left(\sum_{\ell=0}^{N-1} y[m]W_N^{\ell(m-n)}\right)$$

$$= \sum_{n=0}^{N-1} x[n] W_N^{-nk} \left(\sum_{\ell=0}^{N-1} y[m] W_N^{\ell m} \right) = X^*[k] Y[k].$$

5.33 Note $X[k]$ is the MN–point DFT of the sequence $x_e[n]$ obtained from $x[n]$ by appending it with $M(N-1)$ zeros. Thus, the length-MN sequence $y[n]$ is given by $y[n] = \sum_{\ell=0}^{M-1} x_e[\langle n - N\ell \rangle_{MN}],\ 0 \le n \le MN-1.$ Taking the MN–DFT of both sides we get $Y[k] = \left(\sum_{\ell=0}^{M-1} W_{MN}^{Nk\ell} \right) X[k] = \left(\sum_{\ell=0}^{M-1} W_N^{k\ell} \right) X[k].$

5.34 **(a)** $X[0] = \sum_{n=0}^{9} x[n] = 30.$

(b) $X[5] = \sum_{n=0}^{9} (-1)^n x[n] = 0.$

(c) $\sum_{k=0}^{9} X[k] = 10 \cdot x[0] = -30.$

(d) The inverse DFT of $e^{-j2\pi k/5} X[k]$ is $x[\langle n-2 \rangle_{10}]$. Thus, $\sum_{k=0}^{9} e^{-j2\pi k/5} X[k]$ $= 10 \cdot x[\langle 0-2 \rangle_{10}] = 10 \cdot x[8] = -100.$

(e) From Parseval's relation, $\sum_{k=0}^{9} |X[k]|^2 = 10 \cdot \sum_{n=0}^{9} |x[n]|^2 = 38600.$

5.35 $X[7] = X^*[\langle -7 \rangle_{12}] = X^*[5] = 2 - j,\ \ X[8] = X^*[\langle -8 \rangle_{12}] = X^*[4] = -3 - j2,$
$X[9] = X^*[\langle -9 \rangle_{12}] = X^*[3] = 6 - j3,\ \ X[10] = X^*[\langle -10 \rangle_{12}] = X^*[2] = 1 + j12,$
$X[11] = X^*[\langle -11 \rangle_{12}] = X^*[1] = 8 + j2.$

(a) $x[0] = \frac{1}{12} \sum_{k=0}^{11} X[k] = 4.5,$

(b) $x[6] = \frac{1}{12} \sum_{k=0}^{11} (-1)^k X[k] = -0.8333,$

(c) $\sum_{n=0}^{11} x[n] = X[0] = 11,$

(d) Let $g[n] = e^{j2\pi n/3} x[n] = W_{12}^{-4n} x[n].$ Then, $\text{DFT}\{W_{12}^{-4n} x[n]\} = X[\langle k-4 \rangle_{12}].$ Thus, $\sum_{n=0}^{11} g[n] = \sum_{n=0}^{11} e^{j2\pi n/3} x[n] = X[\langle 0-4 \rangle_{12}] = X[8] = -3 - j2,$

(e) From Parseval's relation, $\sum_{n=0}^{11} |x[n]|^2 = \frac{1}{12} \sum_{k=0}^{11} |X[k]|^2 = 74.8333.$

5.36 Now, $y_C[n] = \sum_{\ell=0}^{6} g[\ell]h[\langle n-\ell\rangle_7]$. Hence,

$$y_C[0] = g[0]h[0] + g[1]h[6] + g[2]h[5] + g[3]h[4] + g[4]h[3] + g[5]h[2] + g[6]h[1],$$
$$y_C[1] = g[0]h[1] + g[1]h[0] + g[2]h[6] + g[3]h[5] + g[4]h[4] + g[5]h[3] + g[6]h[2],$$
$$y_C[2] = g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[6] + g[4]h[5] + g[5]h[4] + g[6]h[3],$$
$$y_C[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0] + g[4]h[6] + g[5]h[5] + g[6]h[4],$$
$$y_C[4] = g[0]h[4] + g[1]h[3] + g[2]h[2] + g[3]h[1] + g[4]h[0] + g[5]h[6] + g[6]h[5],$$
$$y_C[5] = g[0]h[5] + g[1]h[4] + g[2]h[3] + g[3]h[2] + g[4]h[1] + g[5]h[0] + g[6]h[6],$$
$$y_C[6] = g[0]h[6] + g[1]h[5] + g[2]h[4] + g[3]h[3] + g[4]h[2] + g[5]h[1] + g[6]h[0].$$

Likewise, $y_L[n] = \sum_{\ell=0}^{6} g[\ell]h[n-\ell]$. Hence,

$$y_L[0] = g[0]h[0],$$
$$y_L[1] = g[0]h[1] + g[1]h[0],$$
$$y_L[2] = g[0]h[2] + g[1]h[1] + g[2]h[0],$$
$$y_L[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0],$$
$$y_L[4] = g[0]h[4] + g[1]h[3] + g[2]h[2] + g[3]h[1] + g[4]h[0],$$
$$y_L[5] = g[0]h[5] + g[1]h[4] + g[2]h[3] + g[3]h[2] + g[4]h[1] + g[5]h[0],$$
$$y_L[6] = g[0]h[6] + g[1]h[5] + g[2]h[4] + g[3]h[3] + g[4]h[2] + g[5]h[1] + g[6]h[0],$$
$$y_L[7] = g[1]h[6] + g[2]h[5] + g[3]h[4] + g[4]h[3] + g[5]h[2] + g[6]h[1],$$
$$y_L[8] = g[2]h[6] + g[3]h[5] + g[4]h[4] + g[5]h[3] + g[6]h[2],$$
$$y_L[9] = g[3]h[6] + g[4]h[5] + g[5]h[4] + g[6]h[3],$$
$$y_L[10] = g[4]h[6] + g[5]h[5] + g[6]h[4],$$
$$y_L[11] = g[5]h[6] + g[6]h[5],$$
$$y_L[12] = g[6]h[6].$$

Comparing $y_C[n]$ with $y_L[n]$ we observe that

$$y_C[0] = y_L[0] + y_L[7],$$
$$y_C[1] = y_L[1] + y_L[8],$$
$$y_C[2] = y_L[2] + y_L[9],$$
$$y_C[3] = y_L[3] + y_L[10],$$
$$y_C[4] = y_L[4] + y_L[11],$$
$$y_C[5] = y_L[5] + y_L[12],$$
$$y_C[6] = y_L[6].$$

5.37 Since $x[n]$ is a length-9 real sequence, its DFT satisfies $X[k] = X^*[\langle -k\rangle_9]$. Therefore,

$$X[1] = X^*[\langle -1\rangle_9] = X^*[8] = -7.7 + j3.2,$$

$X[3] = X^*[\langle -3\rangle_9] = X^*[6] = 8.6 + j9.6,$
$X[5] = X^*[\langle -5\rangle_9] = X^*[4] = -3.5 - j5.3,$
$X[7] = X^*[\langle -7\rangle_9] = X^*[2] = 1.2 + j4.1.$

5.38 $X[1] = X^*[\langle -1\rangle_9] = X^*[8] = 4.5 - j1.6,$
$X[4] = X^*[\langle -4\rangle_9] = X^*[5] = -3.1 - j8.2,$
$X[6] = X^*[\langle -6\rangle_9] = X^*[3] = -7.2 + 4.1,$
$X[7] = X^*[\langle -7\rangle_9] = X^*[2] = 1.2 + j2.3.$

5.39 Since the DFT $X[k]$ is real-valued, $x[n]$ is a circularly even sequence, i.e.,
$x[n] = x[\langle -n\rangle_{12}]$. Therefore,
$x[1] = x[\langle -1\rangle_{12}] = x[11] = -2,$
$x[4] = x[\langle -4\rangle_{12}] = x[8] = 9.3,$
$x[7] = x[\langle -7\rangle_{12}] = x[5] = 4.1,$
$x[9] = x[\langle -9\rangle_{12}] = x[3] = -3.25,$
$x[10] = x[\langle -10\rangle_{12}] = x[2] = 0.7.$

5.40 Since the DFT $X[k]$ is imaginary-valued, $x[n]$ is a circularly odd sequence, i.e.,
$x[n] = -x[\langle -n\rangle_{12}]$. Therefore,
$x[7] = -x[\langle -7\rangle_{12}] = -x[5] = 9.3,$
$x[8] = -x[\langle -8\rangle_{12}] = -x[4] = -2.87,$
$x[9] = -x[\langle -9\rangle_{12}] = -x[3] = -4.1,$
$x[10] = -x[\langle -10\rangle_{12}] = -x[2] = 3.25,$
$x[11] = -x[\langle -11\rangle_{12}] = -x[1] = -0.7.$

5.41 $X[k] = X^*[\langle -k\rangle_{174}] = X^*[174 - k].$
$X[9] = X^*[174 - 9] = X^*[165] = -3.4 + j5.9 \Rightarrow X[165] = -3.4 - j5.9.$
$X[51] = X^*[174 - 51] = X^*[123] = 5 - j1.6 \Rightarrow X[123] = 5 + j1.6.$
$X[113] = X^*[174 - 113] = X^*[61] = 8.7 - j4.9 \Rightarrow X[61] = 8.7 + j4.9.$
$X[162] = X^*[174 - 162] = X^*[12] = 7.1 - j2.4 \Rightarrow X[12] = 7.1 + j2.4.$
$X[k_1] = 7.1 + j2.4,\ X[k_2] = 8.7 + j4.9,\ X[k_3] = 5 + j1.6,\ X[k_4] = -3.4 - j5.9.$

(a) Comparing these 4 DFT samples with the DFT samples given above we conclude
$k_1 = 12,\ k_2 = 61,\ k_3 = 123,\ k_4 = 165.$

(b) dc value of $\{x[n]\} = X[0] = 11.$

(c)
$$x[n] = \frac{1}{174}\sum_{k=0}^{173} X[k]W_{174}^{-kn} = \frac{1}{174}\Big(X[0] + 2\,\mathrm{Re}\{X[9]W_{174}^{-9n}\} + 2\,\mathrm{Re}\{X[51]W_{174}^{-51n}\}$$
$$+ X[87]W_{174}^{-87n} + 2\,\mathrm{Re}\{X[113]W_{174}^{-113n}\} + 2\,\mathrm{Re}\{X[162]W_{174}^{-162n}\}\Big)$$

(d) $\sum_{n=0}^{173} |x[n]|^2 = \frac{1}{174} \sum_{k=0}^{173} |X[k]|^2 = 86.0279.$

5.42 $X[k] = X^*[\langle -k \rangle_{126}] = X^*[126-k].$

$X[0] = 12.8 + j\alpha.$

$X[13] = X^*[126-13] = X^*[113] = -3.7 + j2.2 \;\Rightarrow X[113] = -3.7 - j2.2.$

$X[51] = X^*[126-51] = X^*[75] = -j1.7 \;\Rightarrow X[75] = j1.7.$

$X[13] = X^*[126-13] = X^*[113] = -3.7 + j2.2 \;\Rightarrow X[113] = -3.7 - j2.2.$

$X[k_1] = X^*[126-k_1] = 9.1 - j5.4 \;\Rightarrow X[126-k_1] = 9.1 + j5.4.$

$X[k_2] = X^*[126-k_2] = 6.3 + j2.3 \;\Rightarrow X[126-k_2] = 6.3 - j2.3.$

$X[k_3] = X^*[126-k_3] = \gamma + j1.7 \;\Rightarrow X[126-k_3] = \gamma - j1.7.$

$X[k_4] = X^*[126-k_4] = -3.7 - j2.2 \;\Rightarrow X[126-k_4] = -3.7 + j2.2.$

(a) (b) Since $x[n]$ is a real-valued sequence of length 126, $X[0]$ and $X[63]$ must be real. Thus, $\alpha = 0$ and $\beta = 0$. As $X[126-k_1]$ and $X[108]$ have the same imaginary part, $\varepsilon = 9.1$ and $k_1 = 126 - 108 = 18$. Likewise, as $X[k_2]$ and $X[79]$ have the same real part, $\delta = -2.3$ and $k_2 = 126 - 79 = 47$. Similarly, as $X[126-k_3]$ and $X[51]$ have the same imaginary part, $\gamma = 0$ and $k_3 = 126 - 51 = 75$. Finally, as $X[k_4] = X[113]$, $k_4 = 113$.

(c) dc value of $x[n]$ is $X[0] = 12.8$.

(d) $x[n] = \frac{1}{126} \sum_{k=0}^{125} X[k] W_{126}^{-kn} = \frac{1}{126}\Big(X[0] + 2\,\mathrm{Re}\{X[13] W_{126}^{-26\pi n/126}\}$

$+ 2\,\mathrm{Re}\{X[18] W_{126}^{-36\pi n/126}\} + 2\,\mathrm{Re}\{X[47] W_{126}^{-94\pi n/126}\} + X[63] W_{126}^{-63\pi n}$

$+ 2\,\mathrm{Re}\{X[75] W_{126}^{-150\pi n/126}\} + 2\,\mathrm{Re}\{X[113] W_{126}^{-226\pi n/126}\}\Big).$

(e) $\sum_{n=0}^{125} |x[n]|^2 = \frac{1}{126} \sum_{k=0}^{125} |X[k]|^2 = 5.767.$

5.43 $Y[k] = W_3^{-2k} X[k] = W_9^{-6k} X[k].$ Therefore, $y[n] = x[\langle n-6 \rangle_9].$ Thus,

$y[0] = x[3] = 4,\; y[1] = x[4] = -3,\; y[2] = x[5] = 5,\; y[3] = x[6] = -2,\; y[4] = x[7] = -2,$

$y[5] = x[8] = 4,\; y[6] = x[0] = 3,\; y[7] = x[1] = 5,\; y[8] = x[2] = 1.$

5.44 $H[k] = H^*[\langle -k \rangle_9] = X^*[9-k].$ Hence, $H[5] = H^*[4] = -6.876 - j11.4883,$

$H[6] = H^*[3] = -j8.6603,\; H[7] = H^*[2] = 6.0346 + j1.957,$

$H[8] = H^*[1] = 6.8414 + j6.0572.$

Now $g[n]=e^{j2\pi n/3}h[n]=e^{j6\pi n/9}h[n]=W_9^{-6n}h[n]$. Therefore,
$G[k]=H[\langle k-6\rangle_9]$, $0\le k\le 8$. Thus, $G[0]=H[\langle -6\rangle_9]=H[3]=j8.6603$,
$G[1]=H[\langle 1-6\rangle_9]=H[4]=-6.876-j11.4883$,
$G[2]=H[\langle 2-6\rangle_9]=H[5]=H^*[4]=-6.876+j11.4883$,
$G[3]=H[\langle 3-6\rangle_9]=H[6]=H^*[3]=-j8.6603$,
$G[4]=H[\langle 4-6\rangle_9]=H[7]=H^*[2]=6.0346+j1.957$,
$G[5]=H[\langle 5-6\rangle_9]=H[8]=H^*[1]=6.8414+j.0572$,
$G[6]=H[\langle 6-6\rangle_9]=H[0]=15$,
$G[7]=H[\langle 7-6\rangle_9]=H[1]=6.8414-j6.0572$,
$G[8]=H[\langle 8-6\rangle_9]=H[2]=6.0346-j1.957$.

5.45 **(a)** $y_L[0]=g[0]h[0]=-4$,
$y_L[1]=g[0]h[1]+g[1]h[0]=10$,
$y_L[2]=g[0]h[2]+g[1]h[1]+g[2]h[0]=-6$,
$y_L[3]=g[0]h[3]+g[1]h[2]+g[2]h[1]=8$,
$y_L[4]=g[1]h[3]+g[2]h[2]=7$,
$y_L[5]=g[2]h[3]=-3$.

(b) $y_C[0]=g_e[0]h[0]+g_e[1]h[3]+g_e[2]h[2]+g_e[3]h[1]$
$=g[0]h[0]+g[1]h[3]+g[2]h[2]=3$,
$y_C[1]=g_e[0]h[1]+g_e[1]h[0]+g_e[2]h[3]+g_e[3]h[2]$
$=g[0]h[1]+g[1]h[0]+g[2]h[3]=7$,
$y_C[2]=g_e[0]h[2]+g_e[1]h[1]+g_e[2]h[0]+g_e[3]h[3]$
$=g[0]h[2]+g[1]h[1]+g[2]h[0]=-6$,
$y_C[3]=g_e[0]h[3]+g_e[1]h[2]+g_e[2]h[1]+g_e[3]h[0]$
$=g[0]h[3]+g[1]h[2]+g[2]h[1]=8$.

(c) $$\begin{bmatrix}G_e[0]\\G_e[1]\\G_e[2]\\G_e[3]\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&-j&-1&j\\1&-1&1&-1\\1&j&-1&j\end{bmatrix}\begin{bmatrix}2\\-1\\3\\0\end{bmatrix}=\begin{bmatrix}4\\-1+j\\6\\-1-j\end{bmatrix},$$

$$\begin{bmatrix}H[0]\\H[1]\\H[2]\\H[3]\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&-j&-1&j\\1&-1&1&-1\\1&j&-1&j\end{bmatrix}\begin{bmatrix}-2\\4\\2\\-1\end{bmatrix}=\begin{bmatrix}3\\-4-j5\\-3\\-4+j5\end{bmatrix}.$$

$$\begin{bmatrix}Y_C[0]\\Y_C[1]\\Y_C[2]\\Y_C[3]\end{bmatrix}=\begin{bmatrix}G_e[0]H[0]\\G_e[1]H[1]\\G_e[2]H[2]\\G_e[3]H[3]\end{bmatrix}=\begin{bmatrix}12\\9+j\\-18\\9-j\end{bmatrix}.$$ Therefore

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ y_C[3] \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 12 \\ 9+j \\ -18 \\ 9-j \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ -6 \\ 8 \end{bmatrix}.$$

(d) $g_e[n] = [2, \quad -1, \quad 3, \quad 0, \quad 0, \quad 0]$, $h_e[n] = [-2, \quad 4, \quad 2, \quad -1, \quad 0, \quad 0]$.

$$y_C[0] = g_e[0]h_e[0] + g_e[1]h_e[5] + g_e[2]h_e[4] + g_e[3]h_e[3] + g_e[4]h_e[2] + g_e[5]h_e[1]$$
$$= g[0]h[0] = -4 = y_L[0],$$
$$y_C[1] = g_e[0]h_e[1] + g_e[1]h_e[0] + g_e[2]h_e[5] + g_e[3]h_e[6] + g_e[4]h_e[3] + g_e[5]h_e[2]$$
$$= g[0]h[1] + g[1]h[0] = 10 = y_L[1],$$
$$y_C[2] = g_e[0]h_e[2] + g_e[1]h_e[1] + g_e[2]h_e[0] + g_e[3]h_e[5] + g_e[4]h_e[4] + g_e[5]h_e[3]$$
$$= g[0]h[2] + g[1]h[1] + g[2]h[0] = -6 = y_L[2],$$
$$y_C[3] = g_e[0]h_e[3] + g_e[1]h_e[2] + g_e[2]h_e[1] + g_e[3]h_e[0] + g_e[4]h_e[5] + g_e[5]h_e[4]$$
$$= g[0]h[3] + g[1]h[2] + g[2]h[1] = 8 = y_L[3],$$
$$y_C[4] = g_e[0]h_e[4] + g_e[1]h_e[3] + g_e[2]h_e[2] + g_e[3]h_e[1] + g_e[4]h_e[0] + g_e[5]h_e[5]$$
$$= g[1]h[3] + g[2]h[2] = 7 = y_L[4],$$
$$y_C[5] = g_e[0]h_e[5] + g_e[1]h_e[4] + g_e[2]h_e[3] + g_e[3]h_e[2] + g_e[4]h_e[1] + g_e[5]h_e[0]$$
$$= g[2]h[3] = -3 = y_L[5].$$

5.46 We need to show $g[n] \textcircled{N} h[n] = h[n] \textcircled{N} g[n]$. Let

$$x[n] = g[n] \textcircled{N} h[n] = \sum_{m=0}^{N-1} g[m]h[\langle n-m \rangle_N] \text{ and}$$

$$y[n] = h[n] \textcircled{N} g[n] = \sum_{m=0}^{N-1} h[m]g[\langle n-m \rangle_N] = \sum_{m=0}^{n} h[m]g[n-m] + \sum_{m=n+1}^{N-1} h[m]g[N+n-m]$$
$$= \sum_{m=0}^{n} h[n-m]g[m] + \sum_{m=n+1}^{N-1} h[N+n-m]g[m] = \sum_{m=0}^{N-1} h[\langle n-m \rangle_N]g[m] = x[n].$$

5.47 **(a)** $y[n] = x_1[n] \textcircled{N} x_2[n] = \sum_{m=0}^{N-1} x_1[m]x_2[\langle n-m \rangle_N]$. Thus,

$$\sum_{n=0}^{N-1} y[n] = \sum_{m=0}^{N-1} x_1[m] \sum_{n=0}^{N-1} x[\langle n-m \rangle_N] = \left(\sum_{n=0}^{N-1} x_1[n]\right)\left(\sum_{n=0}^{N-1} x_2[n]\right).$$

(b)

$$\sum_{n=0}^{N-1} (-1)^n y[n] = \sum_{m=0}^{N-1} x_1[m] \sum_{n=0}^{N-1} x[\langle n-m \rangle_N](-1)^n$$

$= \left(\sum_{m=0}^{N-1} x_1[m]\right)\left(\sum_{n=0}^{m-1} x_2[N+m-n](-1)^n + \sum_{n=m}^{N-1} x_2[m-n](-1)^n\right)$. Replacing n by

$N+n-m$ in the first sum on the right-hand side and by $n-m$ in the second sum on the right-hand side we obtain

$$\sum_{n=0}^{N-1} (-1)^n y[n] = \left(\sum_{m=0}^{N-1} x_1[m]\right)\left(\sum_{n=0}^{m-1} x_2[n](-1)^{n-N+m} + \sum_{n=m}^{N-1} x_2[n](-1)^{n+m}\right)$$

$$= \left(\sum_{n=0}^{N-1} (-1)^n x_1[n]\right)\left(\sum_{n=0}^{N-1} (-1)^n x_2[n]\right).$$

5.48 $y[n] = x[3n],\ 0 \le n \le \frac{N}{3}-1$. Therefore, $Y[k] = \sum_{n=}^{\frac{N}{3}-1} y[n] W_{N/3}^{nk} = \sum_{n=}^{\frac{N}{3}-1} x[4n] W_{N/3}^{nk}$.

Now, $x[n] = \frac{1}{N}\sum_{m=0}^{N-1} X[m] W_N^{-3mn} = \frac{1}{N}\sum_{m=0}^{N-1} X[m] W_{N/3}^{-mn}$. Hence,

$Y[k] = \frac{1}{N}\sum_{n=0}^{\frac{N}{3}-1}\sum_{m=0}^{N-1} X[m] W_{N/3}^{-mn} W_{N/3}^{nk} = \frac{1}{N}\sum_{m=0}^{N-1} X[m] \sum_{n=0}^{\frac{N}{3}-1} W_{N/3}^{(k-m)n}$. Since

$\sum_{n=0}^{\frac{N}{3}-1} W_{N/3}^{(k-m)n} = \begin{cases} \frac{N}{3}, & m = k,\ k+\frac{N}{3},\ k+\frac{2N}{3},\ k+N, \\ 0, & \text{elsewhere.} \end{cases}$ Thus,

$$Y[k] = \frac{1}{3}\left(X[k] + X[k+\tfrac{N}{3}] + X[k+\tfrac{2N}{3}] + X[k+N]\right).$$

5.49 $v[n] = x[n] + jy[n]$. Hence, $X[k] = \frac{1}{2}\{V[k] + V^*[\langle -k\rangle_8]\}$ is the 8–point DFT of $x[n]$

and $Y[k] = \frac{1}{2j}\{V[k] - V^*[\langle -k\rangle_8]\}$ is the 8–point DFT of $y[n]$.

$V[k] = [3+j7,\quad -2+j6,\quad 1-j5,\quad 4-j9,\quad 5+j2,\quad 3-j2,\quad j4,\quad -3-j8]$.

$V^*[\langle -k\rangle_8] = [3+j7,\quad -3+j8,\quad -j4,\quad 3+j2,\quad 5-j2,\quad 4+j9,\quad 1+j5,\quad -2-j6]$.

Therefore,

$X[k] = \left[3+j7,\quad -\frac{5}{2}+j7,\quad \frac{1}{2}+j\frac{9}{2},\quad \frac{7}{2}-j\frac{7}{2},\quad 5,\quad \frac{7}{2}+j\frac{7}{2},\quad \frac{1}{2}-j\frac{9}{2},\quad -\frac{5}{2}-j7\right]$,

$Y[k] = \left[0,\quad -1-j\frac{1}{2},\quad -\frac{1}{2}-j\frac{1}{2},\quad -\frac{11}{2}-j\frac{1}{2},\quad 2,\quad -\frac{11}{2}+j\frac{1}{2},\quad -\frac{1}{2}+j\frac{1}{2},\quad -1+j\frac{1}{2}\right]$.

The IDFT of $V[k]$ obtained using MATLAB is given by

```
Columns 1 through 4

   1.3750 - 0.6250i  -0.8044 + 1.7223i  -1.7500 + 1.2500i
-0.9331 - 0.1187i

    Columns 5 through 8

   0.8750 + 2.6250i   2.5544 - 0.2223i   3.5000 + 1.2500i
-1.8169 + 1.1187i
```

The same result is obtained by computing the IDFT $x[n]$ of $X[k]$ and the IDFT $y[n]$ of $Y[k]$ using MATLAB and then forming $x[n]+jy[n]$.

5.50 $v[n]=g[n]+jh[n]=[2-j2, \quad -1+j4, \quad 3+j2, \quad -j]$. Therefore,

$$\begin{bmatrix} V[0] \\ V[1] \\ V[2] \\ V[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2-j2 \\ -1+j4 \\ 3+j2 \\ -j \end{bmatrix} = \begin{bmatrix} 4+j3 \\ 4-j3 \\ 6-j3 \\ -6-j5 \end{bmatrix}, \text{ i.e.,}$$

$V[k]=[4+j3, \quad 4-j3, \quad 6-j3, \quad -6-j5]$.

Thus, $V^*[\langle -k\rangle_4]=[4+j3, \quad -6+j5, \quad 6+j3, \quad 4+j3]$. Therefore,

$G[k]=\frac{1}{2}\left(V[k]+V^*[\langle -k\rangle_4]\right)=[4, \quad -1+j, \quad 6, \quad -1-j]$ and

$H[k]=\frac{1}{2j}\left(V[k]-V^*[\langle -k\rangle_4]\right)=[3, \quad -4-j5, \quad -3, \quad -4+j5]$.

5.51 Let $p[n]=\text{IDFT}\{P[k]\}$. Thus,

$$\begin{bmatrix} p[0] \\ p[1] \\ p[2] \\ p[3] \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} -5 \\ -2+j5 \\ 4 \\ -2-j5 \end{bmatrix} = \begin{bmatrix} -1.25 \\ -4.75 \\ 0.75 \\ 0.25 \end{bmatrix}$$

$$\begin{bmatrix} d[0] \\ d[1] \\ d[2] \\ d[3] \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 3 \\ 4+j \\ -7 \\ 4-j \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}. \text{ Therefore,}$$

$$X(e^{j\omega})=\frac{-1.25-4.75e^{-j\omega}+0.75e^{-j2\omega}+0.25e^{-j3\omega}}{1+2e^{-j\omega}-3e^{-j2\omega}+3e^{-j3\omega}}.$$

5.52 Let $p[n]=\text{IDFT}\{P[k]\}$. Thus,

$$\begin{bmatrix} p[0] \\ p[1] \\ p[2] \\ p[3] \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 8 \\ -5-j6 \\ -3 \\ -5+j6 \end{bmatrix} = \begin{bmatrix} -1.25 \\ 5.75 \\ 3.75 \\ -0.25 \end{bmatrix}$$

$$\begin{bmatrix} d[0] \\ d[1] \\ d[2] \\ d[3] \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 3 \\ 4+j \\ -7 \\ 4-j \end{bmatrix} = \begin{bmatrix} 2.75 \\ -3.75 \\ -3.25 \\ -1.75 \end{bmatrix}. \text{ Therefore,}$$

$$X(e^{j\omega})=\frac{-1.25+54.75e^{-j\omega}+3.75e^{-j2\omega}-0.25e^{-j3\omega}}{2.75-3.75e^{-j\omega}-3.25e^{-j2\omega}-1.75e^{-j3\omega}}.$$

5.53 $X(e^{j\omega})=\sum_{n=0}^{N-1}x[n]e^{-j\omega n}$ and $\hat{X}[k]=\sum_{n=0}^{N-1}x[n]e^{-j2\pi kn/M}$. Now,

$$\hat{x}[n] = \frac{1}{M}\sum_{k=0}^{M-1}\hat{X}[k]W_M^{-nk} = \frac{1}{M}\sum_{k=0}^{M-1}\sum_{m=0}^{N-1}x[m]e^{-j2\pi km/M}W_M^{-nk}$$

$$= \frac{1}{M}\sum_{m=0}^{N-1}x[m]\sum_{k=0}^{N-1}e^{-j2\pi k(m-n)/M} = \sum_{r=-\infty}^{\infty}x[n+rM].$$

Thus, $\hat{x}[n]$ is obtained by shifting $x[n]$ in multiples of M and adding the shifted copies. Since the new sequence is obtained by shifting in multiples of hence, to recover the original sequence take any consecutive N samples in the range $0 \le n \le N-1$ for any value of r. This would be true only if the shifted copies of $x[n]$ did not overlap with each other, that is, if and only if $M \ge N$.

5.54 **(a)** $X(e^{j\omega}) = \sum_{n=0}^{8}x[n]e^{-j\omega n}$. Therefore, $X_1[k] = \sum_{n=0}^{8}x[n]e^{-j2\pi kn/12}$. Hence,

$$x_1[n] = \frac{1}{12}\sum_{k=0}^{11}X[k]e^{j2\pi kn/12} = \frac{1}{12}\sum_{k=0}^{11}\left(\sum_{m=0}^{8}x[m]e^{j2\pi km/12}\right)e^{j2\pi kn/12}$$

$$= \frac{1}{12}\sum_{m=0}^{8}x[m]\sum_{k=0}^{11}e^{j2\pi k(n-m)/12} = \sum_{r=-\infty}^{\infty}x[n+12r]$$ using the results of Problem 5.53.

Since $M = 12$ and $N = 9$, $M > N$, and hence, $x[n]$ is recoverable from $x_1[n]$. In fact $x_1[n] = \{1, \ -2, \ 3, \ -4, \ 5, \ -4, \ 3, \ -2, \ 1, \ 0, \ 0, \ 0\}, 0 \le n \le 11$, and $x[n]$ is given by the first 9 samples of $x_1[n]$.

(b) Here, $X_2[k] = \sum_{n=0}^{8}x[n]e^{-j2\pi kn/8}$. Hence,

$$x_1[n] = \frac{1}{8}\sum_{k=0}^{7}X[k]e^{j2\pi kn/8} = \frac{1}{8}\sum_{k=0}^{7}\left(\sum_{m=0}^{8}x[m]e^{j2\pi km/8}\right)e^{j2\pi kn/8}$$

$$= \frac{1}{8}\sum_{m=0}^{8}x[m]\sum_{k=0}^{7}e^{j2\pi k(n-m)/8} = \sum_{r=-\infty}^{\infty}x[n+8r]$$ using the results of Problem 5.53.

Since $M = 8$ and $N = 9$, $M < N$, and hence, $x[n]$ is not recoverable from $x_1[n]$. In fact $x_2[n] = [2, \ -2, \ 3, \ -4, \ 5, \ -4, \ 3, \ -2, \ 2, \ -1, \ 1], 0 \le n \le 10.$

5.55 $x[n] = \frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-kn}$. Let $n = N-m$. Then,

$$x[m] = \frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-k(N-m)} = \frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{km} = \frac{1}{N}\mathcal{F}\{X[k]\}.$$ Therefore,

$x[N-n] = \frac{1}{N}\mathcal{F}\{X[k]\} = \frac{1}{N}\mathcal{F}\{\mathcal{F}\{x[n]\}\}$, or, $\mathcal{F}\{\mathcal{F}\{x[n]\}\} = N\cdot x[N-n]$. Hence,

$\mathcal{F}\{\mathcal{F}\{\mathcal{F}\{\mathcal{F}\{x[n]\}\}\}\} = N^2\cdot x[N-n].$

5.56 $y[n] = x[n] \circledast h[n] = \sum_{k=0}^{100} x[k]h[n-k] = \sum_{k=0}^{100} h[k]x[n-k] = \sum_{k=17}^{36} h[k]x[n-k].$

$u[n] = x[n] \textcircled{N} h[n] = \sum_{k=0}^{50} h[k]x[\langle n-k\rangle_{51}] = \sum_{k=17}^{36} h[k]x[\langle n-k\rangle_{51}].$

Now, for $n \geq 36$, $x[\langle n-k\rangle_{51}] = x[n-k]$. Thus, $y[n] = u[n]$ for $36 \leq n \leq 50$.

5.57 **(a)** Overlap-add method: Since the impulse response is of length and the DFT size to be used is 128, hence, the number of data samples required for each convolution will be $128 - 109 = 19$. Thus the total number of DFTs required for the length-1300 data sequence is $\left\lceil \frac{1300}{19} \right\rceil = 69$. Also, the DFT of the impulse response needs to be computed once. Hence, the total number of DFTs used are $= 69 + 1 = 70$. The total number of IDFTs used are $= 69$.

(b) Overlap-save method: In this case, since the first $110 - 1 = 109$ points are lost, we need to pad the data sequence with 109 zeros for a total length of 1409. Again, each convolution will result in $128 - 109 = 19$ correct values. Thus the total number of DFTs required for the data are $\left\lceil \frac{1409}{19} \right\rceil = 75$. Again, 1 DFT is required for the impulse response. The total number of DFTs used are $75 + 1 = 76$. The total number of IDFTs used are $= 75$.

5.58 **(a)** $y[n] = \begin{cases} x[n/L], & n = 0, L, 2L, \ldots, (N-1)L, \\ 0, & \text{elsewhere.} \end{cases}$

$Y[k] = \sum_{n=0}^{NL-1} y[n]W_{NL}^{nk} = \sum_{n=0}^{N-1} x[n]W_{NL}^{nLk} = \sum_{n=0}^{N-1} x[n]W_N^{nk}$. For $k \geq N$, let $k = k_o + rN$

where $k_o = \langle k\rangle_N$. Then, $Y[k] = Y[k_o + rN] = \sum_{n=0}^{N-1} x[n]W_N^{n(k_o+rN)} = \sum_{n=0}^{N-1} x[n]W_N^{nk_o}$

$= X[k_o] = X[\langle k\rangle_N]$.

(b) Since $Y[k] = X[\langle k\rangle_5]$ for $0 \leq k \leq 20$, a sketch of $Y[k]$ is thus as shown below.

5.59 $x_0[n] = x[2n+1] + x[2n]$, $x_1[n] = x[2n+1] - x[2n]$, $y_0[n] = y[2n+1] + y[2n]$, and $y_1[n] = y[2n+1] - y[2n]$, $0 \leq n \leq \frac{N}{2} - 1$. Since $x[n]$ and $y[n]$ are real, symmetric sequences, it follows that $x_0[n]$ and $y_0[n]$ are real, symmetric sequences, and $x_1[n]$ and $y_1[n]$ are real, antisymmetric sequences. Now, consider the $\frac{N}{2}$–length sequence $u[n] = x_0[n] + y_1[n] + j(x_1[n] + y_0[n])$. Its conjugate sequence is given by $u^*[n] = x_0[n] + y_1[n] - j(x_1[n] + y_0[n])$. Next, we observe that

$u[\langle -n\rangle_{N/2}] = x_0[\langle -n\rangle_{N/2}] + y_1[\langle -n\rangle_{N/2}] + j(x_1[\langle -n\rangle_{N/2}] + y_0[\langle -n\rangle_{N/2}])$

$= x_0[n] - y_1[n] + j\left(-x_1[n] + y_0[n]\right)$. Its conjugate sequence is given by
$u^*[\langle -n\rangle_{N/2}] = x_0[n] - y_1[n] - j\left(-x_1[n] + y_0[n]\right)$.
By adding the last 4 sequences we get
$4x_0[n] = u[n] + u^*[n] + u[\langle -n\rangle_{N/2}] + u^*[\langle -n\rangle_{N/2}]$.
From Table 5.3, if $U[k] = \text{DFT}\{u[n]\}$, then $U^*[\langle -k\rangle_{N/2}] = \text{DFT}\{u^*[n]\}$,
$U^*[k] = \text{DFT}\{u^*[\langle -n\rangle_{N/2}]\}$, and $U[\langle -k\rangle_{N/2}] = \text{DFT}\{u[\langle -n\rangle_{N/2}]\}$. Thus,
$X_0[k] = \text{DFT}\{x_0[n]\} = \frac{1}{4}\left(U[k] + U^*[\langle -k\rangle_{N/2}] + U[\langle -k\rangle_{N/2}] + U^*[k]\right)$. Similarly,
$j4x_1[n] = u[n] - u^*[n] - u[\langle -n\rangle_{N/2}] + u^*[\langle -n\rangle_{N/2}]$. Hence,
$X_1[k] = \text{DFT}\{x_1[n]\} = \frac{1}{j4}\left(U[k] - U^*[\langle -k\rangle_{N/2}] - U[\langle -k\rangle_{N/2}] + U^*[k]\right)$. Likewise,
$4y_1[n] = u[n] - u[\langle -n\rangle_{N/2}] + u^*[n] - u^*[\langle -n\rangle_{N/2}]$. Thus,
$Y_1[k] = \text{DFT}\{y_1[n]\} = \frac{1}{4}\left(U[k] - U[\langle -k\rangle_{N/2}] + U^*[\langle -k\rangle_{N/2}] - U^*[k]\right)$. Finally,
$j4y_0[n] = u[n] + u[\langle -n\rangle_{N/2}] - u^*[n] - u^*[\langle -n\rangle_{N/2}]$. Hence,
$Y_0[k] = \text{DFT}\{y_0[n]\} = \frac{1}{j4}\left(U[k] + U[\langle -k\rangle_{N/2}] - U^*[\langle -k\rangle_{N/2}] - U^*[k]\right)$.

5.60 $X_{GDFT}[k,a,b] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi(n+a)(k+b)}{N}}$.

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi(n+a)(k+b)}{N}} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{r=0}^{N-1} x[r]\, e^{-j\frac{2\pi(r+a)k+b)}{N}} e^{j\frac{2\pi(n+a)k+b)}{N}}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{r=0}^{N-1} x[r]\, e^{j\frac{2\pi(n+a-r-a)(k+b)}{N}} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{r=0}^{N-1} x[r]\, e^{j\frac{2\pi(n-r)(k+b)}{N}}$$

$$= \frac{1}{N}\sum_{r=0}^{N-1} x[r] \sum_{k=0}^{N-1} e^{j\frac{2\pi(n-r)(k+b)}{N}} = \frac{1}{N}\cdot x[n]\cdot N = x[n], \text{ as from Eq. (5.23)}$$

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi(n-r)(k+b)}{N}} = \begin{cases} N, & \text{if } n = r, \\ 0, & \text{otherwise.} \end{cases}$$

5.61 **(a)** $x[n] = \alpha\, g[n] + \beta\, h[n]$. Thus,

$$X_{\text{DCT}}[k] = \sum_{n=0}^{N-1} x[n]\cos\left(\frac{\pi k(2n+1)}{2N}\right) = \sum_{n=0}^{N-1}\left(\alpha\, g[n] + \beta\, h[n]\right)\cos\left(\frac{\pi k(2n+1)}{2N}\right)$$

$$= \sum_{n=0}^{N-1}\alpha\, g[n]\cos\left(\frac{\pi k(2n+1)}{2N}\right) + \sum_{n=0}^{N-1}\beta\, h[n]\cos\left(\frac{\pi k(2n+1)}{2N}\right) = \alpha\, G_{\text{DCT}}[k] + \beta\, H_{\text{DCT}}[k].$$

(b) $G_{DCT}[k]=\sum_{n=0}^{N-1} g[n]\cos\left(\frac{\pi k(2n+1)}{2N}\right) \Rightarrow G^*_{DCT}[k]=\sum_{n=0}^{N-1} g^*[n]\cos\left(\frac{\pi k(2n+1)}{2N}\right).$

Therefore, $g^*[n] \overset{DCT}{\longleftrightarrow} G^*_{DCT}[k].$

(c) Note that $\sum_{n=0}^{N-1}\cos\left(\frac{\pi k(2n+1)}{2N}\right)\cos\left(\frac{\pi m(2n+1)}{2N}\right)=\begin{cases} N, & \text{if } k=m=0,\\ N/2, & \text{if } k=m \text{ and } k\neq 0,\\ 0, & \text{otherwise.}\end{cases}$ Now,

$$g[n]g^*[n]=\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1}\alpha[k]\alpha[m]G_{DCT}[k]G^*_{DCT}[m]\cos\left(\frac{\pi(2n+1)k}{2N}\right)\cos\left(\frac{\pi(2n+1)k}{2N}\right).$$

Thus,

$$\sum_{n=0}^{N-1}\left|g[n]\right|^2=\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1}\alpha[k]\alpha[m]G_{DCT}[k]G^*_{DCT}[m]\cos\left(\frac{\pi(2n+1)k}{2N}\right)\cos\left(\frac{\pi(2n+1)k}{2N}\right).$$

Using the orthogonality property mentioned earlier we get

$$\sum_{n=0}^{N-1}\left|g[n]\right|^2=\frac{1}{2N}\sum_{k=0}^{N-1}\alpha[k]\left|G_{DCT}[k]\right|^2.$$

5.62 (a) $\mathbf{H}_N=\begin{bmatrix}13 & 13 & 13 & 13\\ 17 & 7 & -7 & -17\\ 13 & -13 & -13 & 13\\ 7 & -17 & 17 & -7\end{bmatrix}$. The matrix $\mathbf{H}_N$ is orthogonal if $\mathbf{H}_N\mathbf{H}_N^T=c\mathbf{I}$

where $\mathbf{I}$ is the 4×4 identity matrix and c is a constant. Now,

$$\mathbf{H}_N\mathbf{H}_N^T=\begin{bmatrix}13 & 13 & 13 & 13\\ 17 & 7 & -7 & -17\\ 13 & -13 & -13 & 13\\ 7 & -17 & 17 & -7\end{bmatrix}\begin{bmatrix}13 & 17 & 13 & 7\\ 13 & 7 & -13 & -17\\ 13 & -7 & -13 & 17\\ 13 & -17 & 13 & -7\end{bmatrix}=\begin{bmatrix}676 & 0 & 0 & 0\\ 0 & 676 & 0 & 0\\ 0 & 0 & 676 & 0\\ 0 & 0 & 0 & 676\end{bmatrix}.$$

Hence, the matrix is orthogonal and all its rows have the same $\mathcal{L}_2$-norm.

(b) $\mathbf{G}_N=\begin{bmatrix}1 & 1 & 1 & 1\\ 2 & 1 & -1 & -2\\ 1 & -1 & -1 & 1\\ 1 & -2 & 2 & -1\end{bmatrix}$. Next, we observe

$$\mathbf{G}_N\mathbf{G}_N^T=\begin{bmatrix}1 & 1 & 1 & 1\\ 2 & 1 & -1 & -2\\ 1 & -1 & -1 & 1\\ 1 & -2 & 2 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & 1 & 1\\ 1 & 1 & -1 & -2\\ 1 & -1 & -1 & -2\\ 1 & -2 & 1 & -1\end{bmatrix}=\begin{bmatrix}4 & 0 & 0 & 0\\ 0 & 10 & 0 & 0\\ 0 & 0 & 4 & 0\\ 0 & 0 & 0 & 10\end{bmatrix}$$ which shows that

the rows of G_N are orthogonal but do not have the same $\mathcal{L}_2$-norms.

5.63 **(a)** $\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Now, $\frac{1}{2}\mathbf{H}_2^t\mathbf{H}_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2$.

$\mathbf{H}_4 = \begin{bmatrix} \mathbf{H}_2 & \mathbf{H}_2 \\ \mathbf{H}_2 & -\mathbf{H}_2 \end{bmatrix}$. Thus,

$$\frac{1}{4}\mathbf{H}_4^t\mathbf{H}_4 = \frac{1}{4}\begin{bmatrix} \mathbf{H}_2^t & \mathbf{H}_2^t \\ \mathbf{H}_2^t & -\mathbf{H}_2^t \end{bmatrix}\begin{bmatrix} \mathbf{H}_2 & \mathbf{H}_2 \\ \mathbf{H}_2 & -\mathbf{H}_2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4\mathbf{I}_2 & 4\mathbf{I}_2 \\ 4\mathbf{I}_2 & 4\mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_4.$$

$$\frac{1}{N}\mathbf{H}_N^t\mathbf{H}_N = \frac{1}{N}\begin{bmatrix} \mathbf{H}_{N/2}^t & \mathbf{H}_{N/2}^t \\ \mathbf{H}_{N/2}^t & -\mathbf{H}_{N/2}^t \end{bmatrix}\begin{bmatrix} \mathbf{H}_{N/2} & \mathbf{H}_{N/2} \\ \mathbf{H}_{N/2} & -\mathbf{H}_{N/2} \end{bmatrix} = \frac{1}{N}\begin{bmatrix} N\mathbf{I}_{N/2} & N\mathbf{I}_{N/2} \\ N\mathbf{I}_{N/2} & N\mathbf{I}_{N/2} \end{bmatrix} = \mathbf{I}_N.$$

(b) From Eq. (5.172), $\sum_{n=0}^{N-1}|x[n]|^2 = \mathbf{x}^t\mathbf{x} = \left(\frac{1}{N}\mathbf{H}_N^t\mathbf{X}_{Haar}\right)^t\left(\frac{1}{N}\mathbf{H}_N^t\mathbf{X}_{Haar}\right)$

$= \frac{1}{N^2}\mathbf{X}_{Haar}^t\mathbf{H}_N\mathbf{H}_N^t\mathbf{X}_{Haar} = \frac{1}{N}\mathbf{X}_{Haar}^t\mathbf{X}_{Haar} = \frac{1}{N}\sum_{k=0}^{N-1}|X_{Haar}[k]|^2$ as from Eq.

(5.171) $\mathbf{H}_N\mathbf{H}_N^t + N$.

5.64 $X_{\text{DHT}}[k] = \sum_{n=0}^{N-1} x[n]\left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right)\right)$. Now,

$X_{\text{DHT}}[k]\left(\cos\left(\frac{2\pi mk}{N}\right) + \sin\left(\frac{2\pi mk}{N}\right)\right)$

$= \sum_{n=0}^{N-1} x[n]\left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right)\right)\left(\cos\left(\frac{2\pi mk}{N}\right) + \sin\left(\frac{2\pi mk}{N}\right)\right)$. Therefore,

$\sum_{k=0}^{N-1} X_{\text{DHT}}[k]\left(\cos\left(\frac{2\pi mk}{N}\right) + \sin\left(\frac{2\pi mk}{N}\right)\right)$

$= \sum_{n=0}^{N-1} x[n] \sum_{k=0}^{N-1}\left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right)\right)\left(\cos\left(\frac{2\pi mk}{N}\right) + \sin\left(\frac{2\pi mk}{N}\right)\right)$.

It can be shown that $\sum_{k=0}^{N-1}\cos\left(\frac{2\pi nk}{N}\right)\cdot\cos\left(\frac{2\pi mk}{N}\right) = \begin{cases} N, & \text{if } m = n = 0, \\ N/2, & \text{if } m = n \neq 0, \\ N/2, & \text{if } m = N - n, \\ 0, & \text{otherwise,} \end{cases}$

$$\sum_{k=0}^{N-1}\sin\left(\frac{2\pi nk}{N}\right)\cdot\sin\left(\frac{2\pi mk}{N}\right) = \begin{cases} N, & \text{if } m = n = 0, \\ N/2, & \text{if } m = n \neq 0, \\ -N/2, & \text{if } m = N - n, \\ 0, & \text{otherwise,} \end{cases}$$

$$\sum_{k=0}^{N-1}\sin\left(\frac{2\pi nk}{N}\right)\cdot\cos\left(\frac{2\pi mk}{N}\right) = \sum_{k=0}^{N-1}\cos\left(\frac{2\pi nk}{N}\right)\cdot\sin\left(\frac{2\pi mk}{N}\right) = 0.$$

Hence, $x[m] = \frac{1}{N}\sum_{n=0}^{N-1} X_{\text{DHT}}[n]\left(\cos\left(\frac{2\pi mk}{N}\right) + \sin\left(\frac{2\pi mk}{N}\right)\right)$.

5.65 **(a)** $y[n] = x[\langle n - n_o \rangle_N] = \begin{cases} x[n - n_o + N], & 0 \le n \le n_o - 1, \\ x[n - n_o], & n_o \le n \le N-1. \end{cases}$

$$Y_{DHT}[k] = \sum_{n=0}^{N-1} y[n]\left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right)\right)$$

$$= \sum_{n=0}^{n_o-1} x[n - n_o + N]\left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right)\right) + \sum_{n=n_o}^{N-1} x[n - n_o]\left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right)\right).$$

Replacing $n - n_o + N$ by n in the first sum and $n - n_o$ by n in the second sum we get

$$Y_{DHT}[k] = \sum_{n=N-n_o}^{N-1} x[n]\left(\cos\left(\frac{2\pi(n+n_o)k}{N}\right) + \sin\left(\frac{2\pi(n+n_o)k}{N}\right)\right)$$

$$+ \sum_{n=0}^{n_o-1} x[n]\left(\cos\left(\frac{2\pi(n+n_o)k}{N}\right) + \sin\left(\frac{2\pi(n+n_o)k}{N}\right)\right)$$

$$= \sum_{n=0}^{N-1} x[n]\left(\cos\left(\frac{2\pi(n+n_o)k}{N}\right) + \sin\left(\frac{2\pi(n+n_o)k}{N}\right)\right)$$

$$= \cos\left(\frac{2\pi n_o k}{N}\right)\sum_{n=0}^{N-1} x[n]\left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right)\right)$$

$$+ \sin\left(\frac{2\pi n_o k}{N}\right)\sum_{n=0}^{N-1} x[n]\left(\cos\left(\frac{2\pi nk}{N}\right) - \sin\left(\frac{2\pi nk}{N}\right)\right)$$

$$= \cos\left(\frac{2\pi n_o k}{N}\right)X_{DHT}[k] + \sin\left(\frac{2\pi n_o k}{N}\right)X_{DHT}[-k].$$

(b) The N–point DHT of $x[\langle -n \rangle_N]$ is $X_{DHT}[-k]$.

(c) $$\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N^2}\sum_{k=0}^{N\text{-}1}\sum_{\ell=0}^{N-1} X_{DHT}[k]X_{DHT}[\ell]\times$$

$$\left(\sum_{n=0}^{N-1}\left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right)\right)\left(\cos\left(\frac{2\pi n\ell}{N}\right) + \sin\left(\frac{2\pi n\ell}{N}\right)\right)\right).$$

Using the orthogonality property, we observe that the above product is equal to N if $k = \ell$ and is equal to zero if $k \ne \ell$. Hence, $\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left(X_{DHT}[k]\right)^2.$

5.66 $\cos\left(\frac{2\pi nk}{N}\right) = \frac{1}{2}\left(W_N^{-nk} + W_N^{nk}\right)$ and $\sin\left(\frac{2\pi nk}{N}\right) = \frac{1}{2j}\left(W_N^{-nk} - W_N^{nk}\right)$ Therefore,

$$X_{DHT}[k] = \frac{1}{2}\sum_{n=0}^{N-1} x[n]\left(W_N^{-nk} + W_N^{nk} - jW_N^{-nk} + jW_N^{nk}\right)$$

$$= \frac{1}{2}\left(X[N-k] + X[k] - jX[N-k] + jX[k]\right).$$

5.67 $y[n] = x[n] \circledN g[n]$. Thus, $Y_{DHT}[k] = \sum_{n=0}^{N-1} y[n]\left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right)\right)$

$$= \sum_{r=0}^{N-1} x[r] \sum_{n=0}^{N-1} g[\langle n-r\rangle_N]\left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right)\right).$$

From the results of Problem 5.65 we have

$$Y_{DHT}[k] = \sum_{\ell=0}^{N-1} x[\ell]\left(G_{DHT}[k]\cos\left(\frac{2\pi\ell k}{N}\right) + G_{DHT}[\langle -k\rangle_N]\sin\left(\frac{2\pi\ell k}{N}\right)\right)$$

$$= G_{DHT}[k]\sum_{\ell=0}^{N-1} x[\ell]\cos\left(\frac{2\pi\ell k}{N}\right) + G_{DHT}[\langle -k\rangle_N]\sum_{\ell=0}^{N-1} x[\ell]\sin\left(\frac{2\pi\ell k}{N}\right)$$

$$= \frac{1}{2}G_{DHT}[k]\left(X_{DHT}[k] + X_{DHT}[\langle -k\rangle_N]\right) + \frac{1}{2}G_{DHT}[\langle -k\rangle_N]\left(X_{DHT}[k] - X_{DHT}[\langle -k\rangle_N]\right)$$

$$= \frac{1}{2}X_{DHT}[k]\left(G_{DHT}[k] + G_{DHT}[\langle -k\rangle_N]\right) + \frac{1}{2}X_{DHT}[\langle -k\rangle_N]\left(G_{DHT}[k] - G_{DHT}[\langle -k\rangle_N]\right).$$

5.68 **(a)** $y[n] = \sum_{k=0}^{N-1}(\beta_1 W_N^{-nk} + \beta_2 W_N^{nk})X_{DCFT}[k]$

$$= \sum_{m=0}^{N-1} x[m] \sum_{k=0}^{N-1}(\alpha_1\beta_1 W_N^{(m-n)k} + \alpha_2\beta_2 W_N^{-(m+n)k} + \alpha_1\beta_2 W_N^{(m+n)k} + \alpha_1\beta_2 W_N^{-(m-n)k})$$

$$= \sum_{m=0}^{N-1} x[m] N\left((\alpha_1\beta_1 + \alpha_2\beta_2)\delta[m-n] + (\alpha_2\beta_1 + \alpha_1\beta_2)(\delta[m+n] + \delta[m+n-N])\right)$$

$$= N\left[(\alpha_1\beta_1 + \alpha_2\beta_2)x[n] + (\alpha_2\beta_1 + \alpha_1\beta_2)\left(x[n]\delta[n] + x[N-n](1-\delta[n])\right)\right].$$

If we require

$$y[n] = x[n],\ 0 \le n \le N-1,$$

then the following conditions must be satisfied:

$$\alpha_2\beta_1 + \alpha_1\beta_2 = 0, \qquad (5\text{-}1)$$

and

$$N(\alpha_1\beta_1 + \alpha_2\beta_2) = 1. \qquad (5\text{-}2)$$

(b) Let $\alpha_1^2 - \alpha_2^2 \ne 0$. Solving for β_1 and β_2 in Eqs. (5-1) and (5-2) we arrive at

$$\beta_1 = \frac{\alpha_1}{N(\alpha_1^2 - \alpha_2^2)}, \quad \beta_2 = \frac{-\alpha_2}{N(\alpha_1^2 - \alpha_2^2)}.$$

Then, the inverse DCFT is given by

$$x[n] = \frac{1}{N(\alpha_1^2 - \alpha_2^2)} \sum_{k=0}^{N-1}(\alpha_1 W_N^{-nk} - \alpha_2 W_N^{nk})X_{DCFT}[k],\ 0 \le n \le N-1.$$

(c) If $\alpha_1 = \alpha_2^* = \alpha_{re} + j\alpha_{im}$ with $\alpha_{re} \ne 0$ and $\alpha_{im} \ne 0$, the expression for $X_{DCFT}[k]$ reduces to

$$X_{\text{DCFT}}[k] = \sum_{n=0}^{N-1}\left(2\alpha_{\text{re}}\cos\left(\frac{2\pi nk}{N}\right) + 2\alpha_{\text{im}}\sin\left(\frac{2\pi nk}{N}\right)\right)x[n]$$

which is real. The inverse DCFT is then given by

$$x[n] = \frac{1}{4N\alpha_1\alpha_2}\sum_{k=0}^{N-1}\left(2\alpha_{\text{im}}\cos\left(\frac{2\pi nk}{N}\right) + 2\alpha_{\text{re}}\sin\left(\frac{2\pi nk}{N}\right)\right)X_{\text{DCFT}}[k].$$

(d) It can be easily shown that the discrete Hartley transform (DHT) of Eq. (5.192) is a special case of the real DCFT with $\alpha_{\text{re}} = \alpha_{\text{im}} = \frac{1}{2N}$.

5.69 (a) $\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\mathbf{H}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$, and

$$\mathbf{H}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

(b) From the structure of $\mathbf{H}_2$, $\mathbf{H}_4$, and $\mathbf{H}_8$, it can be seen that $\mathbf{H}_4 = \begin{bmatrix} \mathbf{H}_2 & \mathbf{H}_2 \\ \mathbf{H}_2 & -\mathbf{H}_2 \end{bmatrix}$, and $\mathbf{H}_8 = \begin{bmatrix} \mathbf{H}_4 & \mathbf{H}_4 \\ \mathbf{H}_4 & -\mathbf{H}_4 \end{bmatrix}$.

(c) $\mathbf{X}_{HT} = \mathbf{H}_N\mathbf{x}$. Therefore, $\mathbf{x} = \mathbf{H}_N^{-1}\mathbf{X}_{HT} = N\cdot\mathbf{H}_N^T\mathbf{X}_{HT} = N\cdot\mathbf{H}_N^*\mathbf{X}_{HT}$. Hence,

$$x[n] = \sum_{k=0}^{N-1}\mathbf{X}_{HT}[k](-1)^{\sum_{i=0}^{\ell-1} b_i(n)b_i(k)},$$

where $b_i(r)$ is the i–th bit in the binary representation of r.

M5.1 (a)
```
N = input('The value of N = ');
k = -N:N;
y = ones(1,2*N+1);
w = 0:2*pi/255:2*pi;
Y = freqz(y, 1, w);
Ydft = fft(y);
n = 0:1:2*N;
plot(w/pi,abs(Y),n*2/(2*N+1),abs(Ydft),'o');
xlabel('\omega/\pi'),ylabel('Amplitude');
```

(b) Replace the statement `k = -N:N;` with `k = 0:N;`, `y = ones(1,2*N+1);` with `y = ones(1,N+1);` the statement `n = 0:1:2*N;` with `n = 0:N;`, and the statement `plot(w/pi,abs(Y),n*2/(2*N+1),abs(Ydft),'o');` with `plot(w/pi,abs(Y),n*2/(N+1),abs(Ydft),'o');`, in the program of Part (a).

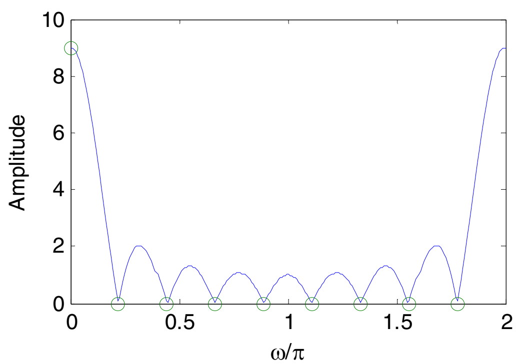
(c) Add the statement `y = y - abs(k)/N;` below the statement `y = ones([1,2*N+1]);` in the program of Part (a).

(d) Replace the statement `y = ones(1,2*N+1);` with `y = N + ones(1,2*N+1) - abs(k);` in the program of Part (a).

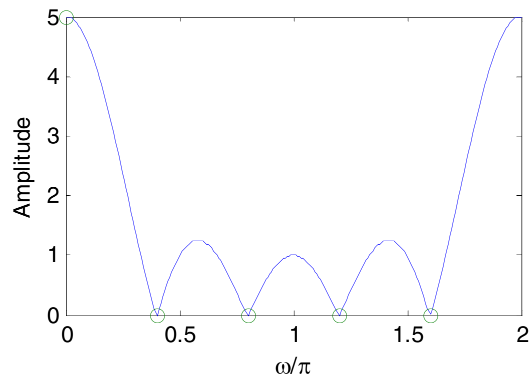
(e) Replace the statement `y = ones(1,2*N+1);` with `y = cos(pi*k/(2*N));` in the program of Part (a).

The plots generated for $N = 4$ are shown below where the circles denote the DFT samples.

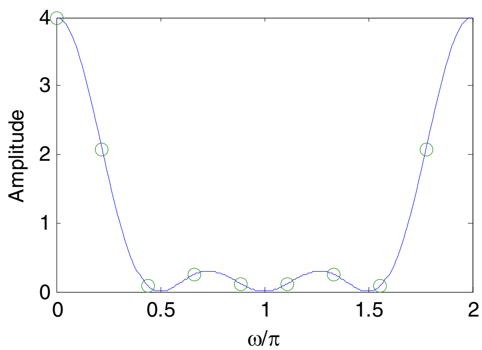
(a)



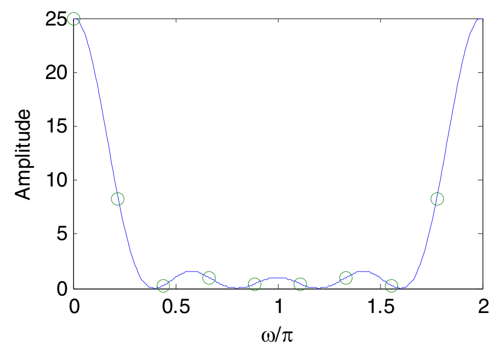
(b)



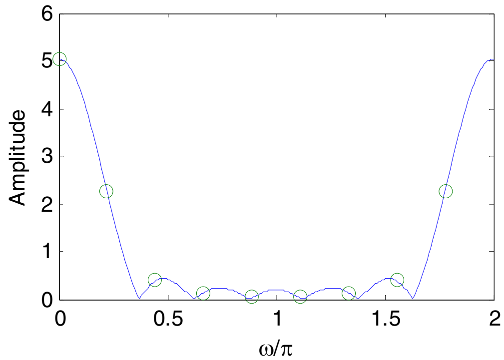
(c)



(d)



(e)



M5.2 The code fragments to be used are as follows:

```
Y = fft(g).*fft(h);
y = iift(Y);
```

(a) $y[n] = g[n] ⑥ h[n]$. The output generated using the above code fragments is

```
y =
     -6       9    -16     20     -4     45
```

(b) $w[n] = x[n] ⑤ v[n]$. The output generated using the above code fragments is

```
w =
Columns 1 through 3
11.0000 +25.0000i  -9.0000 +48.0000i   3.0000 +17.0000i

Columns 4 through 5
29.0000 + 0.0000i  -10.0000 +12.0000i
```

(c) $u[n] = x[n] ⑤ y[n]$. The output generated using the above code fragments is

```
u =
  -23.0000  -69.0000   35.0000  105.0000   73.0000
```

M5.3

```
N = 8; % sequence length
gamma = 0.5;
k = 0:N-1;
x = exp(-gamma*k);
X = fft(x);

% Property 1
X1 = fft(conj(x));
G1 = conj([X(1) X(N:-1:2)]);
% Verify X1 = G1

% Property 2
x2 = conj([x(1) x(N:-1:2)]);
X2 = fft(x2);
```

```
% Verify X2 = conj(X)

% Property 3
x3 = real(x);
X3 = fft(x3);
G3 = 0.5*(X+conj([X(1) X(N:-1:2)]));
% Verify X3 = G3

% Property 4
x4 = j*imag(x);
X4 = fft(x4);
G4 = 0.5*(X-conj([X(1) X(N:-1:2)]));
% Verify X4 = G4

% Property 5
x5 = 0.5*(x+conj([x(1) x(N:-1:2)]));
X5 = fft(x5);
% Verify X5 = real(X)

% Property 6
x6 = 0.5*(x-conj([x(1) x(N:-1:2)]));
X6 = fft(x6);
% Verify X6 = j*imag(X)
```

M5.4

```
N = 8;
k = 0:N-1;
gamma = 0.5;
x = exp(-gamma*k);
X = fft(x);

% Property 1
xpe = 0.5*(x+[x(1) x(N:-1:2)]);
xpo = 0.5*(x-[x(1) x(N:-1:2)]);
Xpe = fft(xpe);
Xpo = fft(xpo);
% Verify Xpe = real(X) and Xpo = j*imag(X)

% Property 2
X2 = [X(1) X(N:-1:2)];
% Verify X = conj(X2);
% real(X) = real(X2) and imag(X) = -imag(X2)
% abs(X)= abs(X2) and angle(X) = -angle(X2)
```

M5.5

```
N = 8; % N is length of the sequence(s)
gamma = 0.5;
k = 0:N-1;
g = exp(-gamma*k); h = cos(pi*k/N);
G = fft(g); H=fft(h);

% Property 1
```

```
alpha=0.5; beta=0.25;
x1 = alpha*g+beta*h;
X1 = fft(x1);
% Verify X1=alpha*G+beta*H

% Property 2
n0 = N/2; % n0 is the amount of shift
x2 = [g(n0+1:N) g(1:n0)];
X2 = fft(x2);
% Verify X2(k)= exp(-j*k*n0)G(k)

% Property 3
k0 = N/2;
x3 = exp(-j*2*pi*k0*k/N).*g;
X3 = fft(x3);
G3 = [G(k0+1:N) G(1:k0)];
% Verify X3=G3

% Property 4
x4 = G;
X4 = fft(G);
G4 = N*[g(1) g(8:-1:2)]; % This forms N*(g mod(-k))
% Verify X4 = G4;

% Property 5
% To calculate circular convolution between
% g and h use eqn (3.67)
h1 = [h(1) h(N:-1:2)];
T = toeplitz(h',h1);
x5 = T*g';
X5 = fft(x5');
% Verify X5 = G.*H

% Property 6
x6 = g.*h;
X6 = fft(x6);
H1 = [H(1) H(N:-1:2)];
T = toeplitz(H.', H1); % .' is the nonconjugate transpose
G6 = (1/N)*T*G.';
% Verify G6 = X6.'
```

M5.6

```
g = input('Type in first sequence = ');
h = input('Type in second sequence = ');
x = g +i*h;
XF = fft(x);
XFconj = conj(XF);
N = length(g);
YF = zeros(N-1,1)';
for k = 1:N-1;
    YF(k) = XFconj(mod(-k,N)+1);
```

```
end
YF1 = [XFconj(1) YF];
GF = (XF + YF1)/2;
HF = (XF - YF1)/2;
```

M5.7

```
x = [-3 5 45 -15 -9 -19 -8 21 -10 23];
XF = fft(x);
k =0:9; YF = exp(-i*2*pi*k/5).*XF;
output = [XF(1) XF(6) sum(XF) sum(YF)];;
disp(output)
disp(sum(abs(XF).*abs(XF))
```

The output data generated by this program is

```
30     0    -30-0.0000i  -100-0.0000i
38600
```

M5.8

```
X = [11 8-i*2 1-i*12 6+i*3  -3+i*2 2+i  15];
k = 8:12; XF(k)=conj(X(mod(-k+2,12)));
XF = [X XF(8:12)];
x = ifft(XF);
n = 0:11; y = exp(i*2*pi*n/3).*x;
output = [x(1)  x(7)  sum(x)  sum(y)];
disp(output)
disp(x.*x)
```

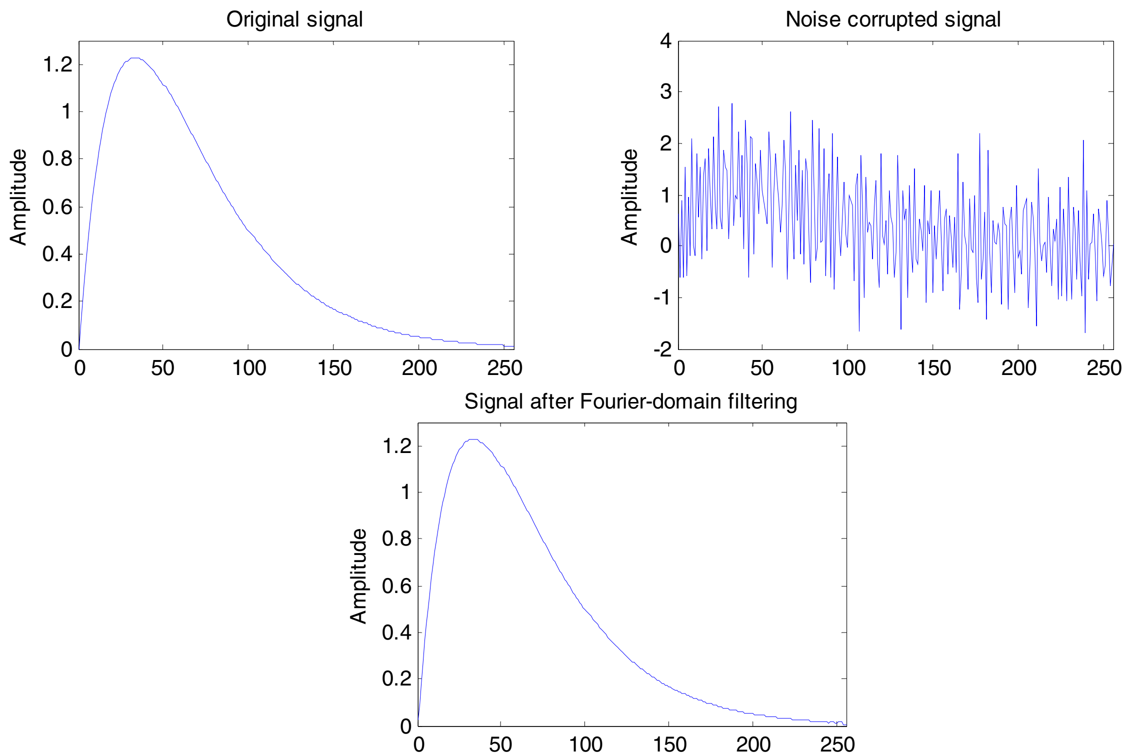
The output data generated by this program is

```
4.5000   -0.8333    11.0000    -3.0000-2.0000i
74.8333
```

M5.9

```
n=0:255;
x = 0.1*n.*exp(-0.03*n);
plot(n,x);axis([0 255 0 1.3]);
xlabel('n');ylabel('Amplitude');
title('Original signal');
pause
z = [zeros(1,50) ones(1,156) zeros(1,50)];
y = 4*rand(1,256)-1;
YF = z.*fft(y);
yinv = ifft(YF);
s = x + yinv;
plot(n,s);axis([0 255 -2 4]);
xlabel('n');ylabel('Amplitude');
title('Noise corrupted signal');
pause
zc = [ones(1,50) zeros(1,156) ones(1,50)];
SF = zc.*fft(s);
xr = ifft(SF);
plot(n,xr);axis([0 255 0 1.3]);
xlabel('n');ylabel('Amplitude');
```

```
title('Signal after Fourier-domain filtering');
```



M5.10

```
function y = overlapsave(x,h)
X = length(x);      %Length of longer sequence
M = length(h);      %length of shorter sequence
flops(0);
if (M > X)          %Error condition
   disp('error');
end
%clear all
temp = ceil(log2(M)); %Find length of circular
convolution
N = 2^temp;            %zero padding the shorter sequence
if(N > M)
    for i = M+1:N
    h(i) = 0;
    end
end
m = ceil((-N/(N-M+1)));
while (m*(N-M+1) <= X)
  if(((N+m*(N-M+1)) <= X)&((m*(N-M+1)) > 0))
      for n = 1:N
         x1(n) = x(n+m*(N-M+1));
      end
  end
if(((m*(N-M+1))<=0)&((N+m*(N-M+1))>=0))
%underflow adjustment
```

```
        for n = 1:N
           x1(n) = 0;
        end
        for n = m*(N-M+1):N+m*(N-M+1)
           if(n > 0)
              x1(n-m*(N-M+1)) = x(n);
           end
        end
   end
   if((N+m*(N-M+1)) > X)  %overflow adjustment
      for n = 1:N
         x1(n) = 0;
      end
       for n = 1:(X-m*(N-M+1))
          x1(n) =x (m*(N-M+1)+n);
       end
   end

      w1 = circonv1(h,x1); %circular convolution using DFT
      for i = 1:M-1
         y1(i) = 0;
      end
      for i = M:N
         y1(i) = w1(i);
      end
      for j = M:N
         if((j+m*(N-M+1)) < (X+M))
            if((j+m*(N-M+1)) > 0)
                yO(j+m*(N-M+1)) = y1(j);
             end
         end
      end
      m = m+1;
   end
   %disp('Convolution using Overlap Save:');
   y = real(yO);

 function y = circonv1(x1,x2)
 L1 = length(x1); L2 = length(x2);
 if L1 ~= L2,
    error('Sequences of unequal lengths'),
 end
 X1 = fft(x1);
 X2 = fft(x2);
 X_RES = X1.*X2;
 y = ifft(X_RES);
```

The MATLAB program for performing convolution using the overlap-save method is

```
h = [1  1  1]/3;
R = 50;
```

```
d = rand(1,R) - 0.5;
m = 0:1:R-1;
s = 2*m.*(0.9.^m);
x = s + d;
%x = [x  x  x  x  x  x  x];
y = overlapsave(x,h);
k = 0:R-1;
plot(k,x,'r-',k,y(1:R),'b--');
xlabel('Time index  n');ylabel('Amplitude');
legend('r-', 's[n]','b--','y[n]');
```

The output plot generated by the above program is shown below: