Digital Communications I: Modulation and Coding Course

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Lecture 3c: Signal Detection in AWGN

Last time we talked about:

- Receiver structure
- Impact of AWGN and ISI on the transmitted signal
- Optimum filter to maximize SNR
  - Matched filter and correlator receiver
- Signal space used for detection
  - Orthogonal N-dimensional space
  - Signal to waveform transformation and vice versa

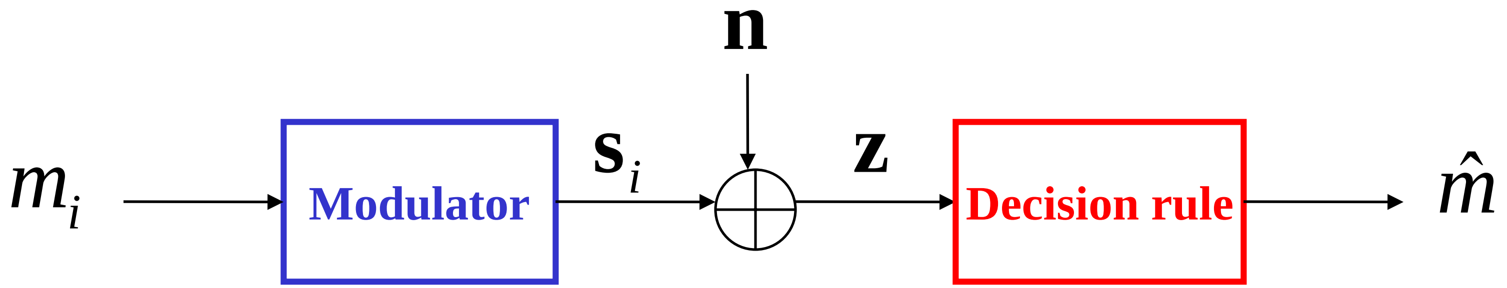
Today we are going to talk about:

- Signal detection in AWGN channels
  - Minimum distance detector
  - Maximum likelihood

- Average probability of symbol error
  - Union bound on error probability
  - Upper bound on error probability based on the minimum distance

Detection of signal in AWGN

- Detection problem:
  - Given the observation vector $\mathbf{z}$, perform a mapping from $\mathbf{z}$ to an estimate $\hat{m}$ of the transmitted symbol, m_i, such that the average probability of error in the decision is minimized.



Statistics of the observation Vector

- AWGN channel model: $\mathbf{z} = \mathbf{s}_i + \mathbf{n}$
  - Signal vector $\mathbf{s}_i = (a_{i1}, a_{i2}, \ldots, a_{iN})$ is deterministic.
  - Elements of noise vector $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ are i.i.d Gaussian random variables with zero-mean and variance $N_0 / 2$. The noise vector pdf is

$$p_{\mathbf{n}}(\mathbf{n}) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|\mathbf{n}\|^2}{N_0} \right)$$

  - The elements of observed vector $\mathbf{z} = (z_1, z_2, \ldots, z_N)$ are independent Gaussian random variables. Its pdf is

$$p_{\mathbf{z}}(\mathbf{z} \mid \mathbf{s}_i) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|\mathbf{z} - \mathbf{s}_i\|^2}{N_0} \right)$$

Detection

- Optimum decision rule (maximum a posteriori probability):

$$\text{Set } \hat{m} = m_i \text{ if}$$
$$\Pr(m_i \text{ sent} \mid \mathbf{z}) \geq \Pr(m_k \text{ sent} \mid \mathbf{z}), \text{ for all } k \neq i$$
$$\text{where } k = 1, \ldots, M.$$

  - Applying Bayes' rule gives:

$$\text{Set } \hat{m} = m_i \text{ if}$$
$$p_k \frac{p_{\mathbf{z}}(\mathbf{z} \mid m_k)}{p_{\mathbf{z}}(\mathbf{z})}, \text{ is maximum for all } k = i$$

Detection …

- Partition the signal space into M decision regions, $Z_1, \ldots, Z_M$ such that

Vector $\mathbf{z}$ lies inside region Z_i if

$$\ln[p_k \frac{p_{\mathbf{z}}(\mathbf{z} \mid m_k)}{p_{\mathbf{z}}(\mathbf{z})}], \text{ is maximum for all } k = i.$$

That means

$$\hat{m} = m_i$$

Detection (ML rule)

- For equal probable symbols, the optimum decision rule (maximum posteriori probability) is simplified to:

$$\text{Set } \hat{m} = m_i \text{ if}$$
$$p_{\mathbf{z}}(\mathbf{z} \mid m_k), \text{ is maximum for all } k = i$$

or equivalently:

$$\text{Set } \hat{m} = m_i \text{ if}$$
$$\ln[p_{\mathbf{z}}(\mathbf{z} \mid m_k)], \text{ is maximum for all } k = i$$

which is known as *maximum likelihood*.

Detection (ML)…

- Partition the signal space into M decision regions, $Z_1, \ldots, Z_M$.

- Restate the maximum likelihood decision rule as follows:

> Vector $\mathbf{z}$ lies inside region Z_i if
>
> $\ln[p_{\mathbf{z}}(\mathbf{z} \mid m_k)]$, is maximum for all $k = i$
>
> That means
>
> $\hat{m} = m_i$

Detection rule (ML)…

- It can be simplified to:

Vector $\mathbf{z}$ lies inside region Z_i if

$$\left\|\mathbf{z}-\mathbf{s}_k\right\|, \text{ is minimum for all } k=i$$

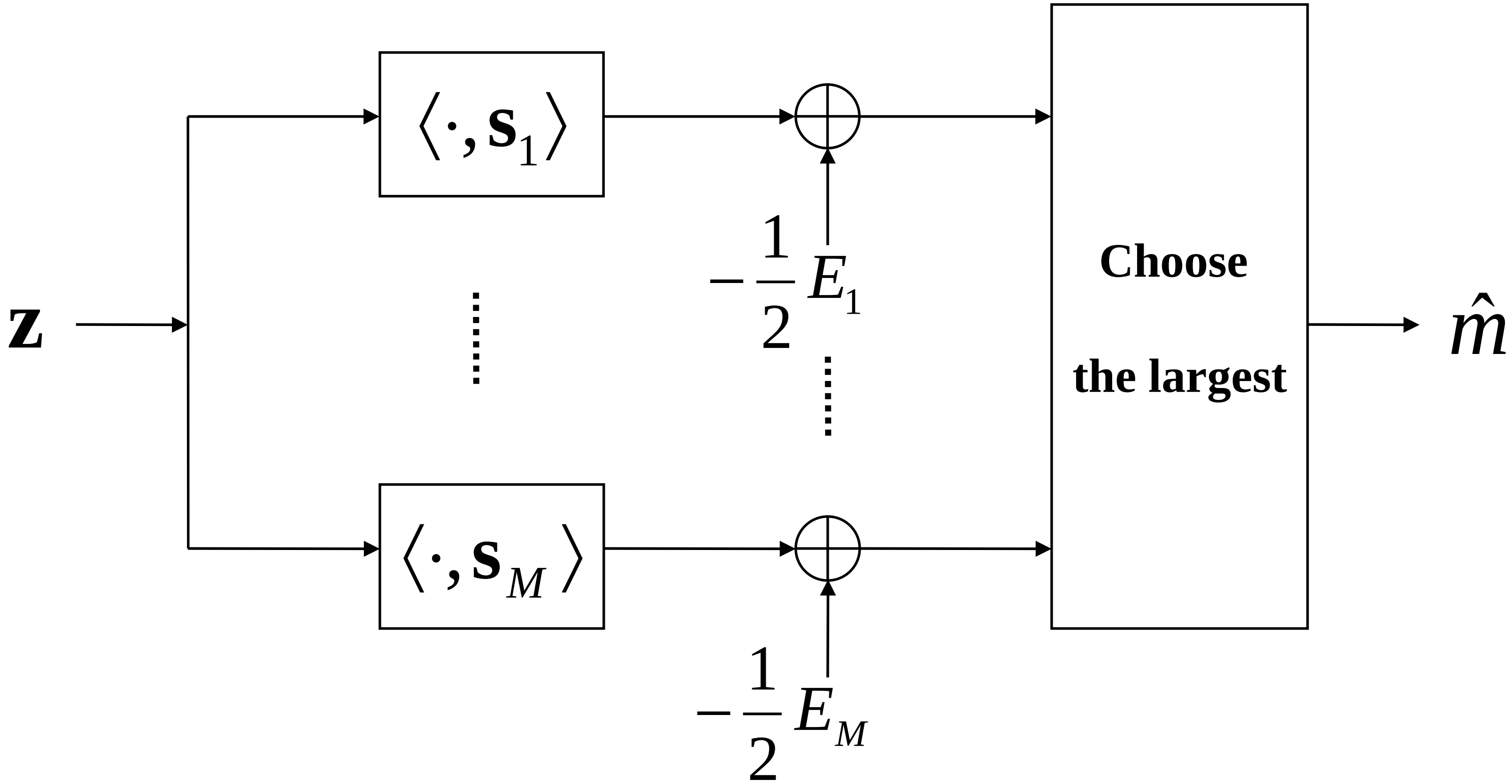
or equivalently:

Vector $\mathbf{r}$ lies inside region Z_i if

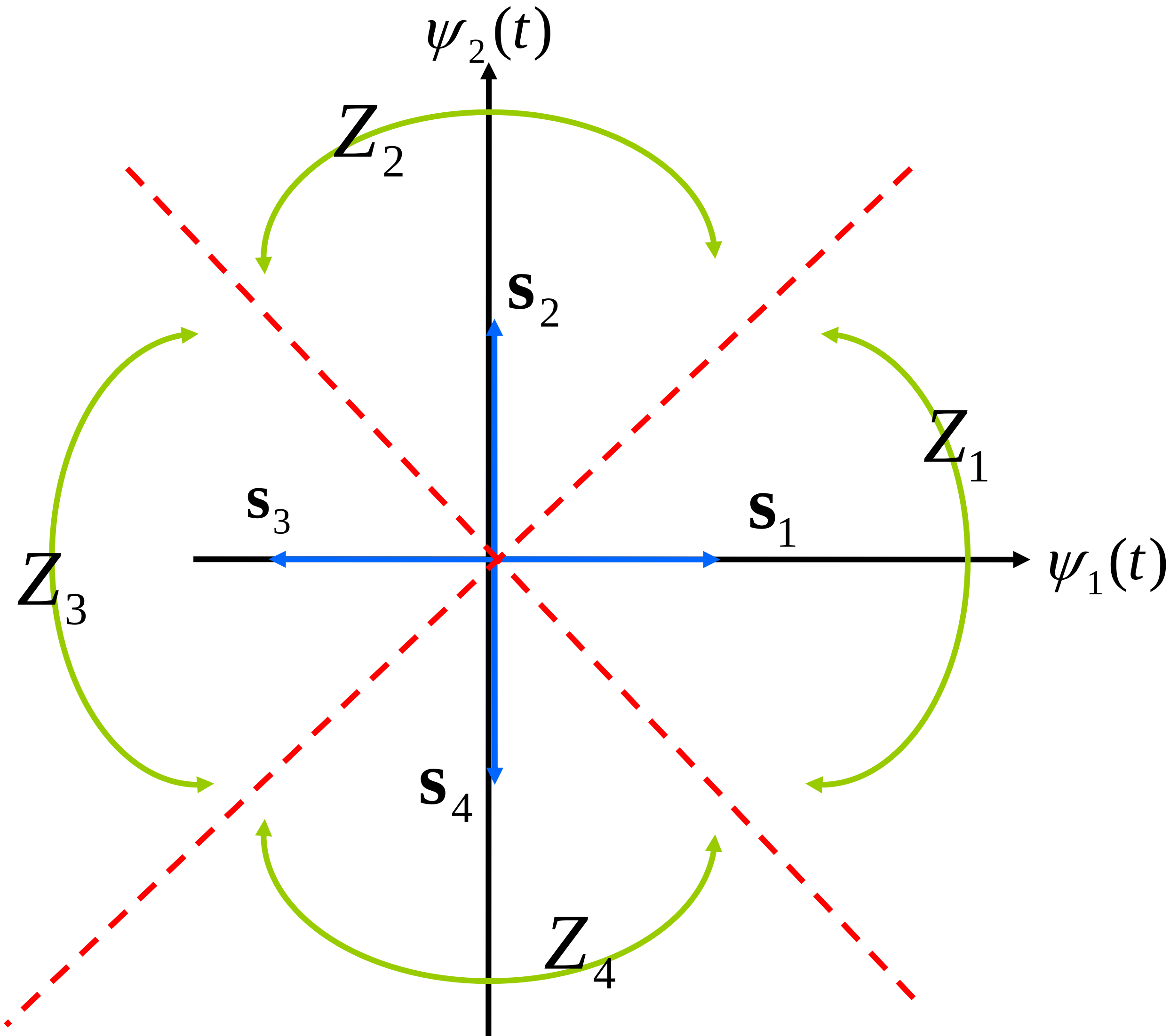
$$\sum_{j=1}^{N} z_j a_{kj} - \frac{1}{2}E_k, \text{ is maximum for all } k=i$$

where E_k is the energy of $s_k(t)$.

Maximum likelihood detector block diagram



Schematic example of the ML decision regions



Average probability of symbol error

- **Erroneous decision:** For the transmitted symbol m_i or equivalently signal vector $\mathbf{s}_i$, an error in decision occurs if the observation vector $\mathbf{z}$ does not fall inside region Z_i.
  - Probability of erroneous decision for a transmitted symbol

$$P_e(m_i) = \Pr(\hat{m} \neq m_i \text{ and } m_i \text{ sent})$$

or equivalently

$$\Pr(\hat{m} \neq m_i) = \Pr(m_i \text{ sent})\Pr(\mathbf{z} \text{ does not lie inside } Z_i \big| m_i \text{ sent})$$

  - Probability of correct decision for a transmitted symbol

$$\Pr(\hat{m} = m_i) = \Pr(m_i \text{ sent})\Pr(\mathbf{z} \text{ lies inside } Z_i \big| m_i \text{ sent})$$

$$P_c(m_i) = \Pr(\mathbf{z} \text{ lies inside } Z_i \big| m_i \text{ sent}) = \int_{Z_i} p_{\mathbf{z}}(\mathbf{z} \,|\, m_i)\, d\mathbf{z}$$

$$P_e(m_i) = 1 - P_c(m_i)$$

Av. prob. of symbol error …

- Average probability of symbol error :

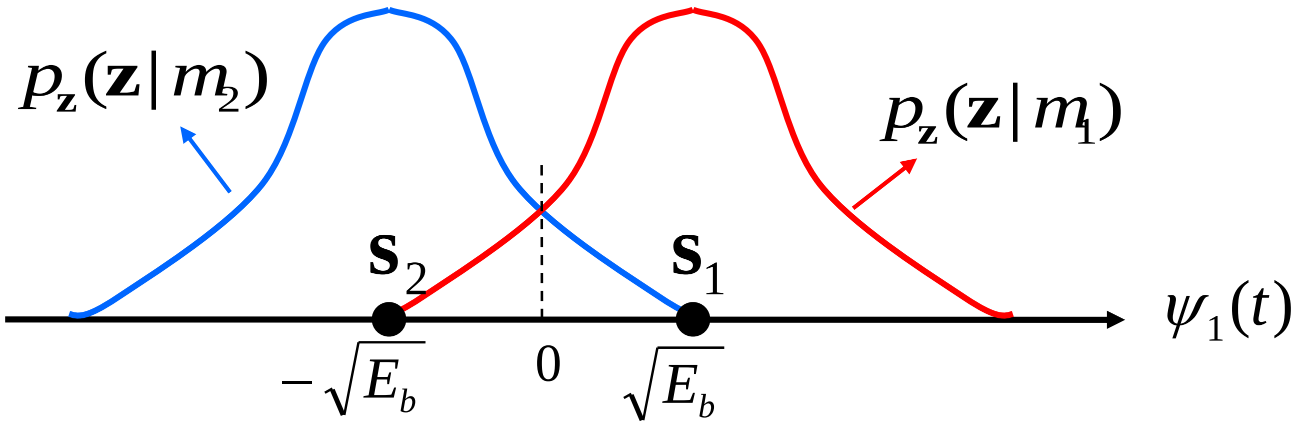
$$P_E(M) = \sum_{i=1}^{M} \Pr(\hat{m} \neq m_i)$$

  - For equally probable symbols:

$$P_E(M) = \frac{1}{M}\sum_{i=1}^{M} P_e(m_i) = 1 - \frac{1}{M}\sum_{i=1}^{M} P_c(m_i)$$
$$= 1 - \frac{1}{M}\sum_{i=1}^{M} \int_{Z_i} p_{\mathbf{z}}(\mathbf{z} \mid m_i)\, d\mathbf{z}$$

Example for binary PAM

This is a poor "artist's conception" of Gaussian curves



$$P_e(m_1) = P_e(m_2) = Q\left(\frac{\|\mathbf{s}_1 - \mathbf{s}_2\|/2}{\sqrt{N_0/2}}\right)$$

$$P_B = P_E(2) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Erfc / Q(x) Table

TABLE B.1 Complementary Error Function $Q(x) = \int_x^\infty (1/\sqrt{2\pi}) \exp(-u^2/2)du$

	$Q(x)$									
x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2168	0.2148
0.8	0.2169	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0094	0.0003
3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002

$$\text{erfc}(x) = 2Q(x\sqrt{2}) \tag{B.20}$$

$$Q(x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{B.21}$$

- This table gives your erfc(x) and is normalized for σ = 1 and mean = 0
- Note that it calculates the area under the Gaussian function from x to ∞ (the tail) where x is a positive number.
- Draw a picture to see which portion of the area under the curve is your interest (eg. The area from 1 to x) and use the table to give you the required area (eg. 0.5 – Q(x)).
- The table can give you 5 digits (the 5th digit is obtained by linear interpolation) to the right of the decimal point. It is a two-dimensional lookup.

Union bound

Union bound

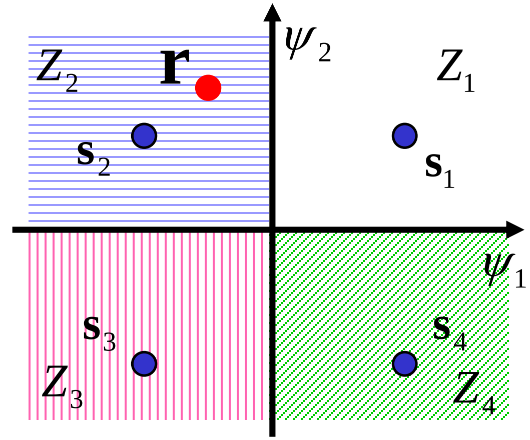
The probability of a finite union of events is upper bounded by the sum of the probabilities of the individual events.

- Let A_{ki} denote that the observation vector $\mathbf{z}$ is closer to the symbol vector $\mathbf{s}_k$ than $\mathbf{s}_i$, when $\mathbf{s}_i$ is transmitted.
- $\Pr(A_{ki}) = P_2(\mathbf{s}_k, \mathbf{s}_i)$ depends only on $\mathbf{s}_i$ and $\mathbf{s}_k$.
- Applying Union bounds yields

$$P_e(m_i) \leq \sum_{\substack{k=1 \\ k \neq i}}^{M} P_2(\mathbf{s}_k, \mathbf{s}_i) \quad \Rightarrow \quad P_E(M) \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{\substack{k=1 \\ k \neq i}}^{M} P_2(\mathbf{s}_k, \mathbf{s}_i)$$

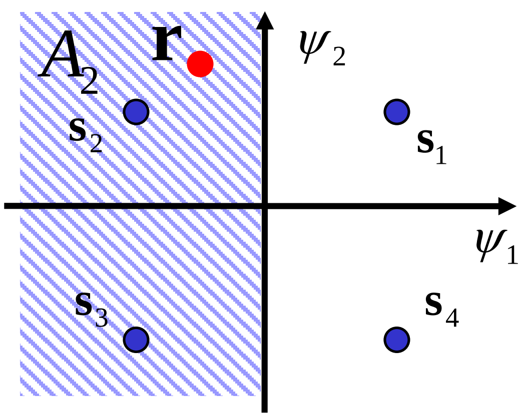
Example of union bound

$$P_e(m_1) = \int_{Z_2 \cup Z_3 \cup Z_4} p_{\mathbf{r}}(\mathbf{r} \mid m_1)\, d\mathbf{r}$$

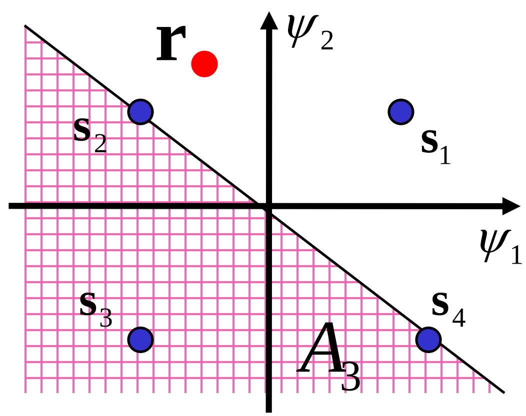


Union bound:

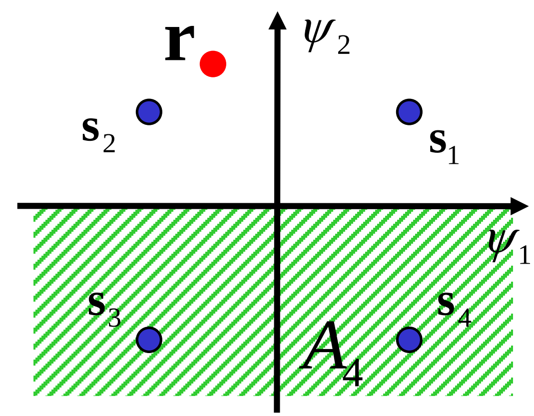
$$P_e(m_1) \le \sum_{k=2}^{4} P_2(\mathbf{s}_k, \mathbf{s}_1)$$



$$P_2(\mathbf{s}_2, \mathbf{s}_1) = \int_{A_2} p_{\mathbf{r}}(\mathbf{r} \mid m_1)\, d\mathbf{r}$$



$$P_2(\mathbf{s}_3, \mathbf{s}_1) = \int_{A_3} p_{\mathbf{r}}(\mathbf{r} \mid m_1)\, d\mathbf{r}$$



$$P_2(\mathbf{s}_4, \mathbf{s}_1) = \int_{A_4} p_{\mathbf{r}}(\mathbf{r} \mid m_1)\, d\mathbf{r}$$

Upper bound based on minimum distance

$$P_2(\mathbf{s}_k, \mathbf{s}_i) = \Pr(\mathbf{z} \text{ is closer to } \mathbf{s}_k \text{ than } \mathbf{s}_i, \text{ when } \mathbf{s}_i \text{ is sent})$$

$$= \int_{d_{ik}}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{u^2}{N_0}\right) du = Q\left(\frac{d_{ik}/2}{\sqrt{N_0/2}}\right)$$

$$d_{ik} = \|\mathbf{s}_i - \mathbf{s}_k\|$$

$$P_E(M) \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{\substack{k=1 \\ k \neq i}}^{M} P_2(\mathbf{s}_k, \mathbf{s}_i) \leq (M-1) Q\left(\frac{d_{\min}/2}{\sqrt{N_0/2}}\right)$$

Minimum distance in the signal space: $d_{\min} = \min\limits_{\substack{i,k \\ i \neq k}} d_{ik}$

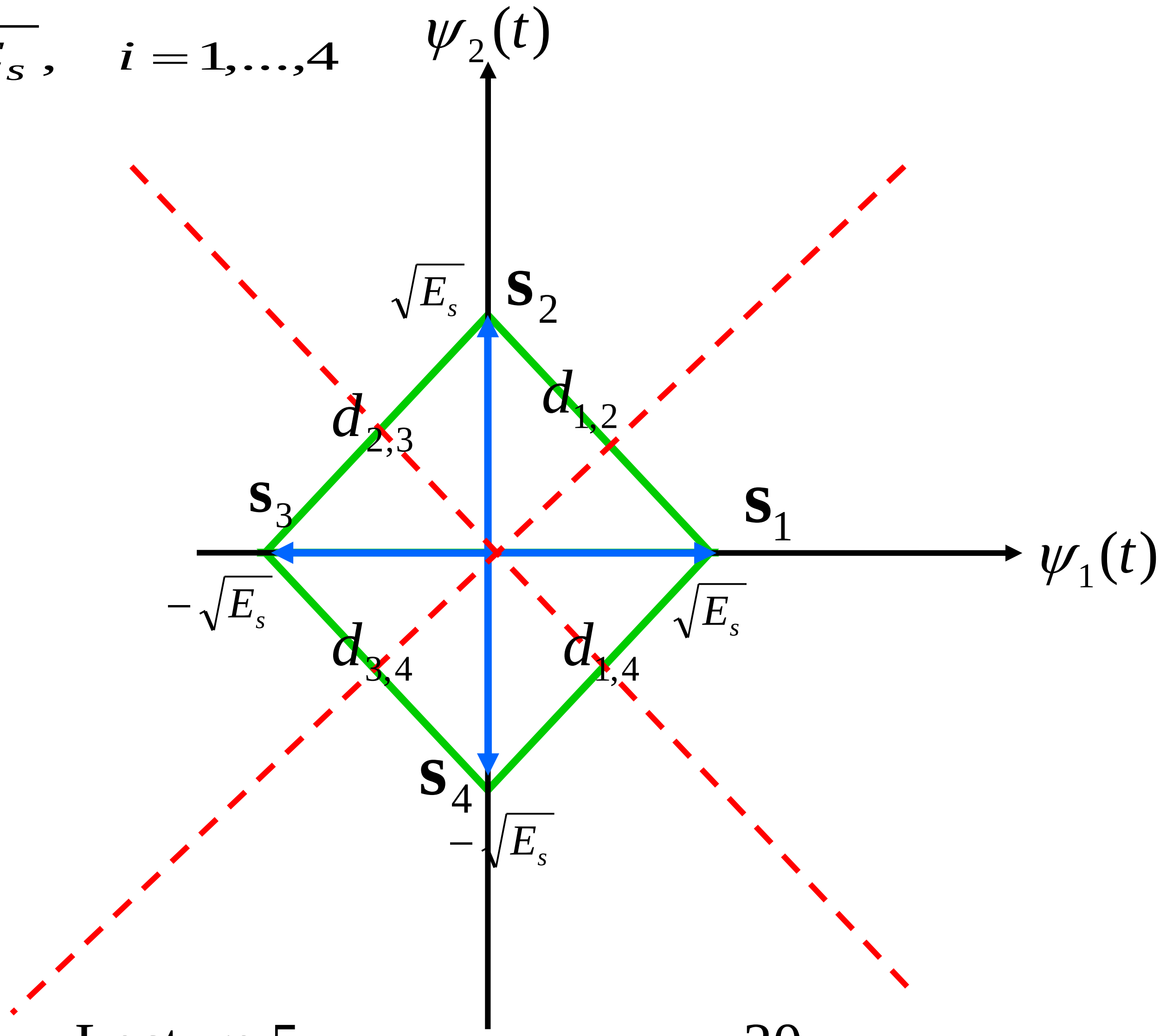
Example of upper bound on av. Symbol error prob. based on union bound

$$\|\mathbf{s}_i\| = \sqrt{E_i} = \sqrt{E_s}, \quad i = 1, \ldots, 4$$

$$d_{i,k} = \sqrt{2E_s}$$

$$i \neq k$$

$$d_{\min} = \sqrt{2E_s}$$



Eb/No figure of merit in digital communications

- SNR or S/N is the average signal power to the average noise power. SNR should be modified in terms of bit-energy in DCS, because:
  - Signals are transmitted within a symbol duration and hence, are energy signal (zero power).
  - A merit at bit-level facilitates comparison of different DCSs transmitting different number of bits per symbol.

$$\frac{E_b}{N_0} = \frac{ST_b}{N/W} = \frac{S}{N}\frac{W}{R_b}$$

R_b : Bit rate

W : Bandwidth

Example of Symbol error prob. For PAM signals