**EE-387 Probability for Electrical and Computer Engineers**
**Solution to Midterm Examination**

**Question 1** (5 points total) Unlike the city of Nirvana, NY where 911 is the all-purpose number for emergencies, in Xond, Matf you dial 01 for a fire emergency, 02 for the police, and 03 for an ambulance. It is estimated that emergency calls in Matf have the same frequency distribution as in Nirvana, namely 60 percent are for the police, 25 percent are for ambulance service, and 15 percent are for the fire department. Assume that 10 calls are monitored and that none of the calls overlap in time.

(a) What is the probability that the ten dialed numbers create the sequence

02030202030102030202.

(b) How many distinguishable sequences exist that involve six calls to the police, three for an ambulance, and one to the fire department?

Solution: (a) $(0.15)^1(0.60)^6(0.25)^3 = 1.0935\times 10^{-4}$. (b) $\frac{10!}{6!3!1!} = 840$.

**Question 2** (5 points total) A smuggler, trying to pass himself off as a glass-bead importer, attempts to smuggle diamonds by mixing diamonds beads among glass beads in the proportion of one diamond bead per 1000 beads. A customs inspector examines a sample of 100 beads. What is the probability that the smuggler will be caught?

Solution: Let $X$ be the random variable that represents the number of diamonds found by the inspector from a sample of 100 beads. It is clear that $X$ has binomial distribution with parameter $n = 100$ and $p = \frac{1}{1000}$.

The probability that the smuggler will not get caught is just the probability that no diamond found by the inspector among a sample of 100 beads, i.e., $P_X(0)$. We have

$$P_X(0) = \binom{100}{0} p^0 (1-p)^{100-0} = (1-p)^{100}.$$

Therefore, the probability that the smuggler gets caught is just $1 - P_X(0) = 1 - (1 - \frac{1}{1000})^{100} =$

0.0952.

**Question 3** (5 points total) A binary communication system has digits "0" and "1" transmitted and digits "0" and "1" received. Denote by $A_1$ and $A_2$ the events that symbols "0" and "1" are received, respectively. Denote by $B_1$ and $B_2$ the events that symbols "0" and "1" are transmitted, respectively. Suppose the channel is very noisy and we have the channel transition probabilities $P[A_1|B_1] = 0.2$, $P[A_2|B_1] = 0.8$, $P[A_1|B_2] = 0.7$, and $P[A_2|B_2] = 0.3$. Assume that the *a priori* symbol transmission probabilities are $P[B_1] = 0.6$ and $P[B_2] = 0.4$.

(a) Find the received symbol probabilities $P[A_1]$ and $P[A_2]$.

(b) Find the *a posteriori* probabilities $P[B_1|A_1]$, $P[B_2|A_1]$, $P[B_1|A_2]$, and $P[B_2|A_2]$.

(c) Determine the decision rule used by a maximum *a posteriori* (MAP) receiver.

Solution: (a) The received symbol probabilities are

$$P[A_1] = P[A_1|B_1]P[B_1] + P[A_1|B_2]P[B_2] = (0.2)(0.6) + (0.7)(0.4) = 0.4.$$

and $P[A_2] = 1 - P[A_1] = 0.6$.

(b) The *a posteriori* probabilities are

$$P[B_1|A_1] = \frac{P[A_1|B_1]P[B_1]}{P[A_1]} = 0.3.$$

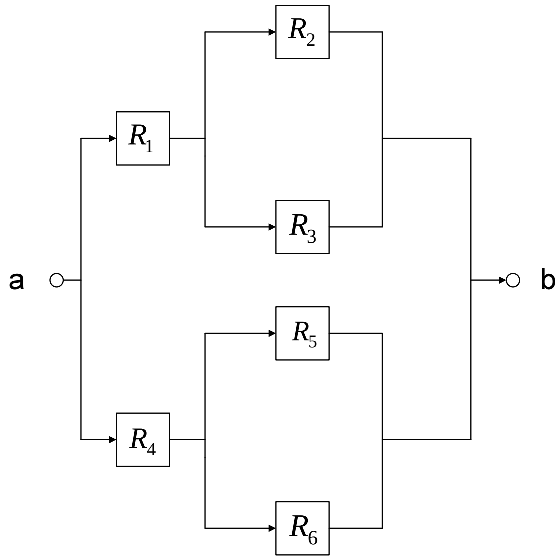
$$P[B_2|A_1] = \frac{P[A_1|B_2]P[B_2]}{P[A_1]} = 0.7.$$

$$P[B_1|A_2] = \frac{P[A_2|B_1]P[B_1]}{P[A_2]} = 0.8.$$

$$P[B_2|A_2] = \frac{P[A_2|B_2]P[B_2]}{P[A_2]} = 0.2.$$

(c) Since $P[B_2|A_1] > P[B_1|A_1]$ and $P[B_1|A_2] > P[B_2|A_2]$, a MAP receiver will decide "1" if "0" is received and decide "0" if "1" is received.

**Question 4** (5 points total) In a communication system the signal sent from point *a* to point *b* passing a series repeaters shown in the following figure. The probabilities of the repeaters failing (independently) are $p_1 = P[R_1] = 0.005$, $p_2 = P[R_2] = P[R_3] = P[R_4] = 0.01$, and $p_3 = P[R_5] = P[R_6] = 0.05$. Find the probability that the signal will not arrive at point *b*.




Solution:

$$\begin{aligned} P[\text{signal will not arrive at b}] &= P[(\text{upper path fails}) \text{ and } (\text{lower path fails})] \\ &= P[\text{upper path fails}]P[\text{lower path fails}] \end{aligned}$$

$$\begin{aligned} P[\text{upper path fails}] &= P[R_1 \cup (R_2 \cap R_3)] \\ &= P[R_1] + P[R_2 \cap R_3] - P[R_1 \cap R_2 \cap R_3] \\ &= P[R_1] + P[R_2]P[R_3] - P[R_1]P[R_2]P[R_3] \\ &= p_1 + p_2^2 - p_1 p_2^2 \\ &= 5.0995 \times 10^{-3}. \end{aligned}$$

Similarly, $P[\text{lower path fails}] = p_2 + p_3^2 - p_2 p_3^2 = 1.2475 \times 10^{-2}$. Therefore $P[\text{signal will not arrive at b}] = (5.0995 \times 10^{-3})(1.2475 \times 10^{-2}) = 6.36163 \times 10^{-5}$.

**Question 5** (5 points total) A biased coin with $P[H] = 1/4$ is tossed 3 times. Let $X$ be the number of heads that appear. (a) Find the probability mass function (PMF) of $X$. (b) Find the cumulative distribution function (CDF) of $X$.

Solution: (a) The PMF of $X$ is

$$P_X(x) = \begin{cases} \binom{3}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^3 & x = 0 \\ \binom{3}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^2 & x = 1 \\ \binom{3}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^1 & x = 2 \\ \binom{3}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^0 & x = 3 \\ 0 & \text{otherwise.} \end{cases}$$

or

$$P_X(x) = \begin{cases} \frac{27}{64} & x = 0 \\ \frac{27}{64} & x = 1 \\ \frac{9}{64} & x = 2 \\ \frac{1}{64} & x = 3 \\ 0 & \text{otherwise.} \end{cases}$$

The CDF of $X$ is

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{27}{64} & 0 \le x < 1 \\ \frac{54}{64} & 1 \le x < 2 \\ \frac{63}{64} & 2 \le x < 3 \\ 1 & 3 \le x. \end{cases}$$

**Question 6** (5 points total) A discrete random variable $X$ has the following probability mass function (PMF)

$$P_X(x) = \begin{cases} 0.2 & x = -2 \\ 0.1 & x = -1 \\ 0.1 & x = 0 \\ 0.3 & x = 1 \\ 0.3 & x = 4 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = X^2$. (a) Find $P_Y(y)$. (b) Find $E[Y]$ and Var[Y].

Solution: (a) Note $S_X = \{-2,-1,0,1,4\}$ and $S_Y = \{0,1,4,16\}$. We have $P_Y(0) = P_X(0) = 0.1$, $P_Y(1) = P_X(-1) + P_X(1) = 0.1 + 0.3 = 0.4$, $P_Y(4) = P_X(-2) = 0.2$, $P_Y(16) = P_X(4) = 0.3$.

Therefore, the PMF of $Y$ is

$$P_Y(y) = \begin{cases} 0.1 & y = 0 \\ 0.4 & y = 1 \\ 0.2 & y = 4 \\ 0.3 & y = 16 \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$E[Y] = \sum_{y \in S_Y} y P_Y(y) = (0)(0.1) + (1)(0.4) + (4)(0.2) + (16)(0.3) = 6.0$$

and

$$E[Y^2] = \sum_{y \in S_Y} y^2 P_Y(y) = (0^2)(0.1) + (1^2)(0.4) + (4^2)(0.2) + (16^2)(0.3) = 80.4.$$

Thus,

$$\text{Var[Y]} = E[Y^2] - (E[Y])^2 = 80.4 - 36.0 = 44.4.$$