LECTURE 4

- **Readings:** Section 1.5
- Review
- Independence of two events
- Independence of a collection of events

Review

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}, \qquad \text{assuming } \mathbf{P}(B) > 0$$

- Multiplication rule:

  $$\mathbf{P}(A \cap B) = \mathbf{P}(B) \cdot \mathbf{P}(A \mid B) = \mathbf{P}(A) \cdot \mathbf{P}(B \mid A)$$

- Total probability theorem:

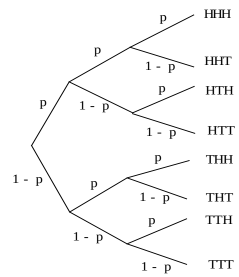
  $$\mathbf{P}(B) = \mathbf{P}(A)\mathbf{P}(B \mid A) + \mathbf{P}(A^c)\mathbf{P}(B \mid A^c)$$

- Bayes rule:

  $$\mathbf{P}(A_i \mid B) = \frac{\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)}{\mathbf{P}(B)}$$

Models based on conditional probabilities

- 3 tosses of a biased coin: $\mathbf{P}(H) = p$, $\mathbf{P}(T) = 1 - p$



$\mathbf{P}(THT) =$

$\mathbf{P}(\text{1 head}) =$

$\mathbf{P}(\text{first toss is H} \mid \text{1 head}) =$

Independence of two events

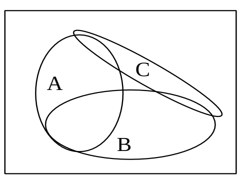
- **Defn:**

  $$\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$$

  - Recall $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B \mid A)$
  - Independence is same as $\mathbf{P}(B \mid A) = \mathbf{P}(B)$ and $\mathbf{P}(A \mid B) = \mathbf{P}(A)$

Conditioning may affect independence

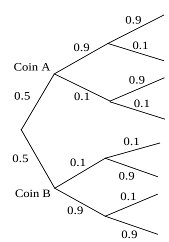
- Assume A and B are independent



- If we are told that C occurred, are A and B independent?

Conditioning may affect independence

- Two unfair coins, A and B: $\mathbf{P}(H \mid \text{coin } A) = 0.9$, $\mathbf{P}(H \mid \text{coin } B) = 0.1$ choose either coin with equal probability



- Once we know it is coin A, are future tosses independent?
- If we do not know which coin it is, are future tosses independent?
  - Compare:
    $\mathbf{P}(\text{toss } 11 = H)$
    $\mathbf{P}(\text{toss } 11 = H \mid \text{first 10 tosses are heads})$

Independence of a collection of events

- Intuitive definition: Information on some of the events tells us nothing about probabilities related to the remaining events
  - E.g.,
    $\mathbf{P}(A_1 \cap (A_2^c \cup A_3) \mid A_5 \cap A_6^c)$
    $= \mathbf{P}(A_1 \cap (A_2^c \cup A_3))$
- Mathematical definition: For any distinct $i, j, \ldots, q$,

  $$\mathbf{P}(A_i \cap A_j \cap \cdots \cap A_q) = \mathbf{P}(A_i)\mathbf{P}(A_j) \cdots \mathbf{P}(A_q)$$

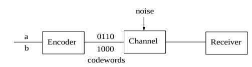
Independence vs. pairwise independence

- Two independent fair coin tosses
  - A: First toss is H
  - B: Second toss is H
  - $\mathbf{P}(A) = \mathbf{P}(B) = 1/2$

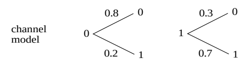
HH	HT
TH	TT

  - C: First and second toss have same outcome
  - $\mathbf{P}(C) =$
  - $\mathbf{P}(C \cap A) =$
  - $\mathbf{P}(A \cap B \cap C) =$
  - $\mathbf{P}(C \mid A \cap B) =$
- Pairwise independence **does not** imply independence

Decoding of noise corrupted messages



- Prior probabilities:
  $\mathbf{P}(a) = 1/3$, $\mathbf{P}(b) = 2/3$

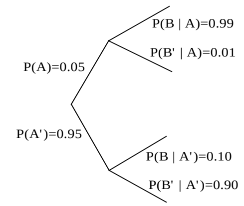


- Received 0001. What was transmitted?

$$\mathbf{P}(a \mid 0001) = \frac{\mathbf{P}(a)\mathbf{P}(0001 \mid a)}{\mathbf{P}(a)\mathbf{P}(0001 \mid a) + \mathbf{P}(b)\mathbf{P}(0001 \mid b)}$$

Radar example from last time

- Event A: Airplane is flying above
  Event B: Something registers on radar screen



$$\begin{aligned} \mathbf{P}(\text{airplane} \mid \text{register}) &= P(A \mid B) \\ &= \frac{\mathbf{P}(A)\mathbf{P}(B \mid A)}{\mathbf{P}(B)} \\ &= \end{aligned}$$