LECTURE 7

- **Readings:** Sections 2.4-2.6

Lecture outline

- Review PMF, expectation, variance
- Conditional PMF
- Geometric PMF
- Total expectation theorem
- Joint PMF of two random variables

Review

- Random variable X: function from sample space to the real numbers
- PMF (for discrete random variables): $p_X(x) = \mathbf{P}(X = x)$
- Expectation:

$$\mathbf{E}[X] = \sum_x x p_X(x)$$

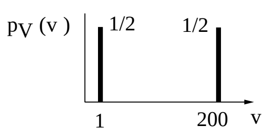
$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x)$$

- $\mathbf{E}[X - \mathbf{E}[X]] =$

$$\begin{aligned} \text{var}(X) &= \mathbf{E}\left[(X - \mathbf{E}[X])^2\right] \\ &= \sum_x (x - \mathbf{E}[X])^2 p_X(x) \\ &= \mathbf{E}[X^2] - (\mathbf{E}[X])^2 \end{aligned}$$

Average speed vs. average time

- Traverse a 200 mile distance at constant but random speed V



- $d = 200$, $T = t(V) = 200/V$

$$\mathbf{E}[V] = 1 \cdot (1/2) + 200 \cdot (1/2) = 100.5$$

$$\begin{aligned} \mathbf{E}[T] = \mathbf{E}[t(V)] &= \sum_v t(v) p_V(v) \\ &= \frac{200}{1} \cdot \frac{1}{2} + \frac{200}{200} \cdot \frac{1}{2} = 100.5 \end{aligned}$$

- $\mathbf{E}[T] \cdot \mathbf{E}[V] \neq 200 = d = \mathbf{E}[TV]$
- $\mathbf{E}[T] \neq 200/\mathbf{E}[V]$.

$$\text{var}(V) = \sum_v (v - \mathbf{E}[V])^2 p_V(v)$$

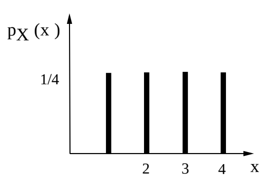
$$= (1-100.5)^2 \frac{1}{2} + (200-100.5)^2 \frac{1}{2} \approx 10,000$$

- standard deviation $\sigma_V = \sqrt{\text{var(V)}} \approx 100$

Conditional expectation

- $p_{X|A}(x) = \mathbf{P}(X = x \mid A)$

$$\mathbf{E}[X \mid A] = \sum_x x p_{X|A}(x)$$



$\mathbf{E}[X \mid X \geq 2] =$

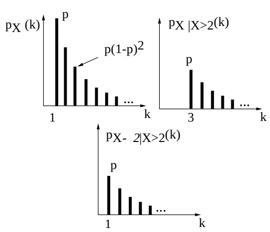
Geometric PMF

- X: number of independent coin tosses until first head

$$p_X(k) = (1-p)^{k-1}p, \qquad k = 1, 2, \dots$$

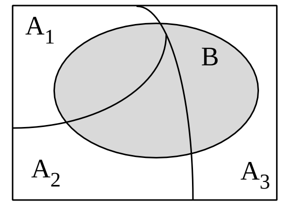
$$\mathbf{E}[X] = \sum_{k=1}^{\infty} k p_X(k) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$$

- Memoryless property: Given that $X > 2$, the r.v. $X - 2$ has same geometric PMF



Total Expectation theorem:

- Partition of sample space into disjoint events $A_1, A_2, \dots, A_n$



$$\mathbf{P}(B) = \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B \mid A_n)$$
$$\mathbf{E}[X] = \mathbf{P}(A_1)\mathbf{E}[X \mid A_1] + \cdots + \mathbf{P}(A_n)\mathbf{E}[X \mid A_n]$$

- Geometric example:
  $A_1: \{X = 1\}, \quad A_2: \{X > 1\}$

$$\begin{aligned}\mathbf{E}[X] \;=\; & \mathbf{P}(X=1)\mathbf{E}[X \mid X=1] \\ & + \mathbf{P}(X>1)\mathbf{E}[X \mid X>1]\end{aligned}$$

- Solve to get $\mathbf{E}[X] = 1/p$

Joint PMFs

- $p_{X,Y}(x,y) = \mathbf{P}(X = x \text{ and } Y = y)$

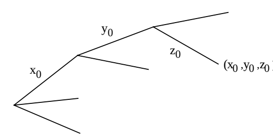
y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

- $\sum_x \sum_y p_{X,Y}(x,y) =$

- $p_X(x) = \sum_y p_{X,Y}(x,y)$

- $p_{X|Y}(x \mid y) = \mathbf{P}(X = x \mid Y = y) = \dfrac{p_{X,Y}(x,y)}{p_Y(y)}$

- $\sum_x p_{X|Y}(x \mid y) =$



$$p_{x,y,z}(x_0, y_0, z_0) = p_x(x_0) \cdot p_{y|x}(y_0 \mid x_0)$$
$$\cdot p_{z|x,y}(z_0 \mid x_0, y_0)$$