LECTURE 9: Continuous R.V.'s

Readings: Sections 3.1–3.3

Lecture Outline

- Probability Density Functions (PDF)
- Cumulative Distribution Functions (CDF)
- Gaussian (Normal) random variables

Probability Density Functions (PDF)

Definition: X is called *continuous* if there exists a non-negative function f_X (PDF of X) such that, for every subset A of the real line

$$\mathsf{P}(X \in A) = \int_A f_X(x)dx$$

Example:

- $\mathsf{P}(c \le X \le d) = \int_c^d f_X(x)dx$
- $\int_{-\infty}^{\infty} f_X(x)dx = 1$
- $\mathsf{P}(x \le X \le x+\delta) \approx f_X(x).\delta$

Means and Variances

Definitions:

- $\mathsf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)dx$
- $\mathsf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)dx$
- $\mathsf{Var}[X] = \sigma_X^2 = \int_{-\infty}^{\infty} (x - \mathsf{E}[X])^2 f_X(x)dx$

Basic Properties:

- Let $Y = \alpha X + \beta$,
  - $\mathsf{E}[Y] = \alpha \mathsf{E}[X] + \beta$
  - $\mathsf{Var}[Y] = \alpha^2 \mathsf{Var}[X]$
- $\mathsf{Var}[X] = \mathsf{E}\left[(X - \mathsf{E}[X])^2\right] = \mathsf{E}\left[X^2\right] - \mathsf{E}[X]^2$

Example: **Uniform**

- $f_X(x) =$ $\qquad a \le x \le b$
- $\mathsf{E}[X] =$
- $\sigma_X^2 = \int_a^b \left(x - \frac{a+b}{2}\right) \frac{1}{b-a} dx = \frac{(b-a)^2}{12}$
- Let $Y = 2 * X + 1$
  - $f_Y(y) =$ $\qquad \le y \le$
  - $\mathsf{E}[Y] =$
  - $\sigma_Y^2 =$

Cumulative Distribution Functions (CDF)

Definition:

$$F_X(x) = \mathsf{P}(X \le x)$$

- Continuous R.V.:

$$F_X(x) = \mathsf{P}(X \le x) = \int_{-\infty}^{x} f_X(t)dt$$

- Discrete R.V.:

$$F_X(x) = \mathsf{P}(X \le x) = \sum_{k \le x} p_X(k)$$

- Mixed Distributions:
  - Schematic drawing of a combination of a PDF and a PMF
  - The corresponding CDF:

$$F_X(x) = \mathsf{P}(X \le x)$$

Gaussian (normal) PDF

Definition: X is said to be a *standard normal* (or *standard Gaussian*) if its PDF is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Notation: $X \sim \mathcal{N}(0, 1)$

$\mathsf{E}[X] =$ $\qquad$ $\mathsf{Var}[X] = 1$

General: X is *normal* (*Gaussian*): $X \sim \mathcal{N}(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$\mathsf{E}[X] = \mu$ $\qquad$ $\mathsf{Var}[X] = \sigma^2$

Property: Let $Y = aX + b$, if $X \sim \mathcal{N}(\mu, \sigma^2)$

- Then $\mathsf{E}[Y] =$ $\qquad$ $\mathsf{Var}[Y] =$
- Fact: $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Calculating normal probabilities

- No closed form available for the CDF

But there are tables
(for the standard normal)

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\dfrac{X - \mu}{\sigma} \sim \mathcal{N}(\quad,\quad)$
- If $X \sim \mathcal{N}(2, 16)$:

  $\mathsf{P}(X \le 3) = \mathsf{P}\left(\dfrac{X-2}{4} \le \dfrac{3-2}{4}\right) = \mathsf{CDF}_S(0.25)$