LECTURE 15

- **Readings:** Sections 4.3, 4.4

Lecture outline

- Conditional expectation
  - Law of iterated expectations
  - Law of conditional variances
- Sum of a random number of independent r.v.'s
  - mean, variance, transform

Conditional expectations

- Given the value y of a r.v. Y:

$$\mathbf{E}[X \mid Y = y] = \sum_x x p_{X|Y}(x \mid y)$$

(integral in continuous case)

- Stick example: stick of length ℓ break at uniformly chosen point Y break again at uniformly chosen point X
- $\mathbf{E}[X \mid Y = y] = \dfrac{y}{2}$ (number)
- $\mathbf{E}[X \mid Y] = \dfrac{Y}{2}$ (r.v.)
- **Law of iterated expectations:** $\mathbf{E}[\mathbf{E}[X \mid Y]] = \sum_y \mathbf{E}[X \mid Y = y] p_Y(y) = \mathbf{E}[X]$
- In stick example: $\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X \mid Y]] = \mathbf{E}[Y/2] = \ell/4$

Conditional variance

- $\text{Var}(X \mid Y)$: variance of the conditional distribution of X

$$\text{var}(X \mid Y = y) = \mathbf{E}\left[(X - \mathbf{E}[X \mid Y = y])^2 \mid Y = y\right]$$

- Interesting formula:

$$\text{Var}(X) = \mathbf{E}[\text{Var}(X \mid Y)] + \text{Var}(\mathbf{E}[X \mid Y])$$

Example

$$\text{Var}(X) = \mathbf{E}[\text{Var}(X \mid Y)] + \text{Var}(\mathbf{E}[X \mid Y])$$

$\mathbf{E}[X \mid Y = 1] =$ $\mathbf{E}[X \mid Y = 2] =$

$\text{Var}(X \mid Y = 1) =$ $\text{Var}(X \mid Y = 2) =$

$\mathbf{E}[X] =$

$\text{Var}(E[X \mid Y]) =$

Sum of a random number of independent r.v.'s

- N: number of stores visited
- X_i: money spent in store i
  - X_i assumed i.i.d.
  - independent of N
- Let $Y = X_1 + \cdots + X_N$

$$\begin{aligned} \mathbf{E}[Y] &= \mathbf{E}[\mathbf{E}[Y \mid N]] \\ &= \mathbf{E}[N\mathbf{E}[X]] \\ &= \mathbf{E}[N]\mathbf{E}[X] \end{aligned}$$

- Variance:

$$\begin{aligned} \mathrm{Var}(Y) &= \mathbf{E}[\mathrm{Var}(Y \mid N)] + \mathrm{Var}(\mathbf{E}[Y \mid N]) \\ &= \mathbf{E}[N]\mathrm{var}(X) + (\mathbf{E}[X])^2\mathrm{var}(N) \end{aligned}$$

Review of transforms

- Definitions:

$$M_X(s) = \mathbf{E}[e^{sX}] = \begin{cases} \sum_x e^{sx} p_X(x) \\ \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx \end{cases}$$

- Moment generating properties:

$$\frac{d^n}{ds^n} M_X(s)\Big|_{s=0} = \mathbf{E}[X^n]$$

- Transform of sum of independent r.v.'s X, Y independent; $W = X + Y$

$$M_W(s) = M_X(s) M_Y(s)$$

- Transform of "random sum":

$$\begin{aligned} M_Y(s) &= \mathbf{E}[e^{sY}] \\ &= \mathbf{E}\left[\mathbf{E}[e^{sY} \mid N]\right] \\ &= \mathbf{E}\left[\mathbf{E}[e^{s(X_1+\cdots+X_N)} \mid N]\right] \\ &= \mathbf{E}\left[M_X(s)^N\right] \end{aligned}$$

  - compare with $M_N(s) = \mathbf{E}[(e^s)^N]$
  - start with $M_N(s)$ and replace occurrences of e^s by $M_X(s)$

Example

$$p_N(n) = \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}, \quad n = 1, 2, \ldots$$

$$f_X(x) = 3e^{-3x}, \qquad x \geq 0$$

- Find pdf of $Y = X_1 + \cdots + X_N$

$$M_N(s) = \frac{e^s/3}{1 - 2e^s/3}, \qquad M_X(s) = \frac{3}{3 - s}$$

$$M_Y(s) = \frac{M_X(s)/3}{1 - 2M_X(s)/3} = \frac{1}{1 - s}$$

$f_Y(y) =$