LECTURE 18

The Poisson process

- **Readings:** Start Section 5.2.

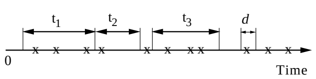
Lecture outline

- Review of Bernoulli process
- Definition of Poisson process
- Distribution of number of arrivals
- Distribution of interarrival times
- Other properties of the Poisson process

Bernoulli review

- Discrete time; success probability p
- Number of arrivals in n time slots: binomial pmf
- Interarrival time pmf: geometric pmf
- Time to k arrivals: Pascal pmf
- Memorylessness

Definition of the Poisson process



- $P(k, \tau) =$ Prob. of k arrivals in interval of duration τ
- Assumptions:
  - Numbers of arrivals in disjoint time intervals are independent
  - For VERY small δ:

$$P(k,\delta) \approx \begin{cases} 1-\lambda\delta & \text{if } k=0 \\ \lambda\delta & \text{if } k=1 \\ 0 & \text{if } k>1 \end{cases}$$

  - $\lambda =$ "arrival rate"

PMF of Number of Arrivals N

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

- $\mathbf{E}[N] = \lambda\tau$
- $\sigma_N^2 = \lambda\tau$
- $M_N(s) = e^{\lambda t(e^s-1)}$

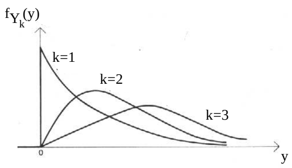
Example: You get email according to a Poisson process at a rate of $\lambda = 0.4$ messages per hour. You check your email every thirty minutes.

- Prob(no new messages)=
- Prob(one new message)=

Interarrival Times

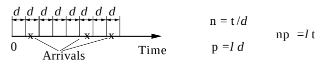
- Y_k time of kth arrival
- Erlang distribution:

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \qquad y \geq 0$$



- First-order interarrival times ($k = 1$): exponential
  $f_{Y_1}(y) = \lambda e^{-\lambda y}, \qquad y \geq 0$
  - Memoryless property: The time to the next arrival is independent of the past

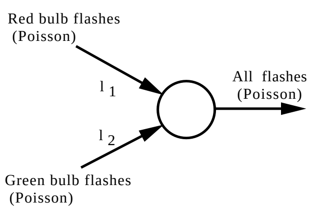
Bernoulli/Poisson Relation



	POISSON	BERNOULLI
Times of Arrival	Continuous	Discrete
Arrival Rate	λ/unit time	p/per trial
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time Distr.	Exponential	Geometric
Time to k-th arrival	Erlang	Pascal

Adding Poisson Processes

- Sum of independent Poisson **random variables** is Poisson
- Sum of independent Poisson **processes** is Poisson



- What is the probability that the next arrival comes from the first process?