332:541 Stochastic Signals and Systems Fall 2007

Pre-Quiz I

You have 20 minutes to complete this short preparatory quizlette. PRINT your full name on THIS side and place your answers and work on BOTH sides of the test sheet. You may use scratch sheets, but ONLY the test sheet will be collected. The point values shown are also suggested time budgets (in minutes) for each problem. This quizlette has a total of 20 points.

1. *(1 point)* Evaluate $\int_{-\infty}^{\infty} f_{X|Y}(x|y) f_Y(y) dy$

   SOLUTION: *The integrand is $f_{XY}(x,y)$ and integrating wrt y gives the marginal $f_X(x)$*

2. *(1 point)* You are given $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$. What is the ratio $f_X(x)/f_Y(y)$?

   SOLUTION:

$$f_{XY}(x,y) = f_{X|Y}(x|y) f_Y(y) = f_{Y|X}(y|x) f_X(x)$$

   so

$$f_X(x)/f_Y(y) = f_{X|Y}(x|y)/f_{Y|X}(y|x)$$

3. *(1 points)* If independent random variables X and Y have Gaussian PDFs with zero means and variances σ_x^2 and σ_y^2 respectively. What is the PDF of the random variable $Z = X - Y$?

   SOLUTION: *$Z = X + (-1)Y$ is a linear superposition, so Z is Gaussian with zero mean and variance $\sigma_x^2 + \sigma_y^2$*

4. *(5 points)* For the previous problem, what is $f_{ZY}(z,y)$?

   SOLUTION: *Z and X and Y are all zero mean jointly Gaussian so all we need are the variances and the correlations (covariances are correlations since zero mean).*

$$E[ZY] = E[(X-Y)Y] = E[XY] - E[Y^2] = -\sigma_y^2$$

$$E[Z^2] = E[(X-Y)^2] = E[X^2] - 2E[XY] + E[Y^2] = \sigma_x^2 + \sigma_y^2$$

   Let random vector $\mathbf{U}$ be

$$\mathbf{U} = \begin{bmatrix} Z \\ Y \end{bmatrix}$$

   with covariance

$$\mathbf{K} = \begin{bmatrix} \sigma_x^2 + \sigma_y^2 & -\sigma_y^2 \\ -\sigma_y^2 & \sigma_y^2 \end{bmatrix}$$

   so

$$f_{\mathbf{U}}(\mathbf{u}) = \frac{1}{2\pi} |\mathbf{K}|^{-1/2} e^{-\frac{1}{2}\mathbf{u}^\top \mathbf{K}^{-1} \mathbf{u}}$$

   We can also expand this out to obtain

$$f_{YZ}(y,z) = \frac{1}{2\pi} \frac{1}{\sigma_x \sigma_y} e^{-\frac{1}{2\sigma_x^2 \sigma_y^2}\left(z^2\sigma_y^2 + 2yz\sigma_y^2 + y^2(\sigma_x^2+\sigma_y^2)\right)}$$

5. *(12 points)* A random variable X is derived from the following experiment:

- Roll a fair k-sided die ($k \geq 2$ a positive integer).
- If side $s \in [1, 2, ..., k]$ turns up, X is chosen from a continuous guniform distribution on $[-s/2, s/2]$.

(a) *(3 points)* Provide an analytic expression and/or carefully labeled sketch for $f_X(x)$?
SOLUTION: *For any given s the distribution on X is uniform on $[-s/2, s/2]$ or*

$$f_{X|S}(x|s) = \frac{1}{s}\left(u(x+s/2) - u(x-s/2)\right)$$

where $u()$ is the unit step function. Thus,

$$f_X(x) = \sum_{s=1}^{k} f_(X|S)(x|s)p_S(s) = \sum_{s=1}^{k} \frac{1}{s}\left(u(x+s/2) - u(x-s/2)\right)\frac{1}{k}$$

which we write as

$$f_X(x) = \frac{1}{k}\sum_{s=1}^{k} \frac{1}{s}\left(u(x+s/2) - u(x-s/2)\right)$$

This looks like a sort of ascending then descending staircase, centered at the origin.

(b) *(1 point)* What is Prob$[X = 0]$?
SOLUTION: *The PDF for X is continuous. The probability of X taking on a particular value is identically zero.*

(c) *(8 points)* Calculate $E[S]$, $E[X]$ and $E[XS]$ where the random variable S is the number of the side which turns up on the die. Are X and S orthogonal, uncorrelated, independent?
SOLUTION: $E[S] = \frac{1}{k}\sum_{s=1}^{k} s = (k+1)/2$. *Each of the conditional distributions of X is zero mean, so we must have have $E[X] = 0$ since it's the weighted sum of conditional means.*

$$E[XS] = \sum_{s} p_S(s)sE[X|S=s] = 0$$

thus X and S are orthogonal. The covariance of X and S is $E[XS] - E[X]E[S] = 0$ so they are uncorrelated too. However, they're obviously not independent since knowing S restricts the possible values of X. For instance, $k = 1$ restricts X to the interval $[-1/2, 1/2]$ while $s = 6$ lets X take on values on $[-3, 3]$.