UNIVERSITY OF CALIFORNIA

College of Engineering

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Professor Tse Fall 1999

EECS 126 — Review Problems

1. A communication channel operates as follows:

At the n-th second, a packet of X_n bits are received at the input of the channel, where X_n is a Poisson random variable of unknown but deterministic rate λ. The random variables X_n are independent.

a) An estimator for λ given $X_1, \ldots, X_n$ is proposed as

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} X_i .$$

Compute the mean square error of this estimator as a function of λ. What happens as $n \to \infty$?

b) Suppose each packet is transmitted with probability $1/2$, independently of all other packets. Let z denote the number of bits transmitted in the first 3 seconds, and assume $\lambda = 1$. Compute the moment generating function

$$M_z(s) = E[e^{sZ}] .$$

c) Let Z be as above. Compute

$$E(Z|X_1) .$$

2. There are two slot machines. To play, you insert one quarter, and the machine returns either no quarter or two quarters. Of the two machines, Machine A returns two quarters with probability 0.4, and Machine B returns two quarters with probability 0.6.

Assume that the machines are memoryless.

a) Calculate the mean of the number of quarters returned by playing a randomly selected machine once.

b) You randomly pick a machine, and play it for 10 times, and win (i.e., it returns two quarters) 6 times. What is the probability that this is Machine B?

c) Suppose you continue to play the machine of Part (b). Given your experience with it, what is the mean of the number of quarters returned by playing it once more?

d) (Think about it after the exam and before going to Las Vegas!) Suppose you are allowed to play 100 times only, what strategy will give you maximum expected return?

3. A class in probability theory is taking a multiple choice test. For a particular problem on the test, the fraction of the examinees who understand the problem is p; $1-p$ is the fraction that will guess. The probability for answering the problem correctly is 1 for an examinee who understands the problem and $1/m$ for one who guesses.

Compute the probability that an examinee understands the problem given that he has correctly answered it. Clearly define your random experiment, the sample space, events, and associated probabilities.

4. Scott's company (Federal Express) delivers 1 million packages a day. The probability that a package will be damaged in the delivery process is 0.01. Assume that product damages occur independently. What is the approximate probability that more than 10,200 packages will be damaged a day? Clearly define the random variables needed for producing the solutions.

5. Let X and Y be independent, identically distributed RVs with the exponential probability density function

$$f_X(w) = f_Y(w) = \lambda e^{-\lambda w} u(w), \quad w \geq 0$$

where $u(\cdot)$ is the unit step function.

a) Determine the probability density function for $R = \begin{cases} \dfrac{X}{X+Y} & X \geq 0 \text{ and } Y > 0 \\ 0 & \text{otherwise} \end{cases}$

b) Determine the conditional density function of R given $X = x$.

6. A multiplexer combines two independent message streams. The number of message arrivals for each message stream is a Poisson random variable with mean 10 message/second. Compute the probability that more than 600 messages arrive at the multiplexer per minute. Compute it explicitly and use approximation if necessary.

7. Compaq buys modems at the price of $50 each. For each delivered batch of modems, Compaq tests the modems for bit errors by randomly picking 10 modems and transmitting through each modem a test message of 10^3 bits.

If the total bit errors occurred while testing 10 modems exceeds 5, Compaq demands a stiff refund of $25 per modem. If the total bit errors do not exceed 5, for each bit error, Compaq demands a refund of $1 per modem (per bit error).

Suppose a type of modem features a bit error probability (the probability of transmitting one bit incorrectly) of 10^{-4}, and suppose bit error occurs independently. What is the average price this type of modem will command from Compaq? (Use appropriate approximation if necessary.)

8. Type A, B, and C items are placed in a common buffer, each type arriving as part of an independent Poisson process with average arrival rates, respectively, of a, b, and c items per minute. For the first four parts of this problem, assume the buffer is discharged immediately whenever it contains a total of ten items.

a) What is the probability that, of the first ten items to arrive at the buffer, only the first *and* one other are type A?

b) What is the probability that any particular discharge of the buffer contains five times as many type A items as type B items?

c) Determine the PDF, expectation, and variance for the total time between consecutive discharges of the buffer.

d) Determine the probability that exactly two of each of the three item types arrive at the buffer input during any particular five minute interval.

For the rest of this problem, a different rule is used for discharging the buffer. For these questions, assume the buffer is discharged immediately whenever it contains a total of three type A items.

e) Determine the PDF, expectation, and variance for the total time between consecutive discharges of the buffer.

f) To an observer arriving at a random time, long after the process began, obtain the PDF's for:

i) U, the time until the arrival of the next item at the buffer input

ii) V, the time until the next discharge of the buffer. (Answer can be obtained without any integrals.)

9. Beginning at time $t = 0$ we begin using bulbs, one at a time, to illuminate a room. Bulbs are replaced immediately upon failure. Each new bulb is selected independently by an equally likely choice between a Type-A bulb and a Type-B bulb.

The lifetime, X, of any particular bulb of a particular type is an independent random variable with the following PDF:

For Type-A Bulbs: $$f_X(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

For Type-B Bulbs: $$f_X(x) = \begin{cases} 3e^{-3x} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Parts I through IV can be addressed independently.

Part I:

a) Find $P(D)$, the probability that there are no bulb failures during the first T hours of this process.

b) Given that there are no failures during the first T hours of this process, determine $P(T_{1A}|D)$, the conditional probability that the first bulb used is a Type-A bulb.

c) Carefully determine the expected value and variance for X, the time until the first bulb failure.

Part II:

d) Find the probability, $P(E)$, that the 12th bulb failure is also the 4th Type-A bulb failure.

e) Up to and including the 12th bulb failure, what is the probability, $P(F)$, that a total of exactly 4 Type-A bulbs have failed?

f) Determine *either* the PDF *or* the transform for U, the time until the 12th bulb failure.

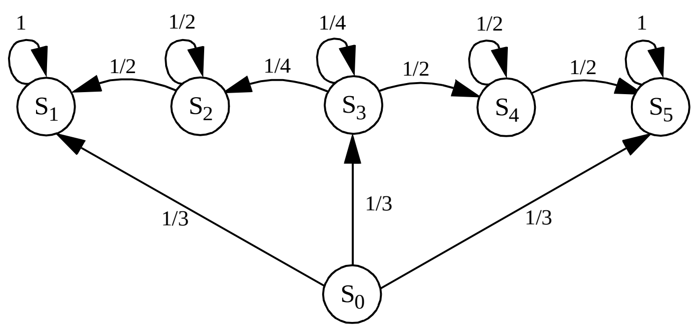
Part III:

g) Determine $P(G)$, the probability that the *total* period of illumination provided by the first *two* Type-B bulbs is longer than that provided by the *first* Type-A bulb.

Part IV:

h) Suppose the process terminates as soon as a total of exactly 12 bulb failures have occurred. Determine the expected value and variance for V, the total period of illumination provided by Type-B bulbs while the process is in operation.

10. For the Markov chain pictured here, the following questions may be answered by inspection:



Given that this process is in state S_0 just before the first trial, determine the probability that:

a) The process enters S_2 for the first time as the result of the K^{th} trial.

b) The process never enters S_4.

c) The process does enter S_2, but it also leaves S_2 on the trial after it entered S_2.

d) The process enters S_1 for the first time on the third trial.

e) The process is in state S_3 immediately after the N^{th} trial.