EE126: Probability and Random Processes **SP'07**

Problem Set 1 — Due Jan, 18

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Problem 1.1. ***Solution***
To write the fraction in the form $a + ib$, we proceed as follows:

$$\begin{aligned} \frac{1+3i}{2+i} &= \frac{1+3i}{2+i} \times \frac{2-i}{2-i} \\ &= \frac{2-i+6i+3}{(2)^2-(i)^2} \\ &= \frac{5+5i}{4+1} = 1+i \end{aligned}$$

To write it in the form $r \times e^{i\theta}$ we notice that

$$\begin{aligned} 1+i &= \sqrt{2}(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) \\ &= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\right) \\ &= \sqrt{2}e^{i\frac{\pi}{4}} \end{aligned}$$

Problem 1.2. ***Solution***
We first verify that

$$\sum_{k=1}^{1} k^3 = 1 = \left(\sum_{k=1}^{1} k\right)^2$$

Thus the equality is true for n=1.
Now assume that it is true for $n-1$ i.e.

$$\sum_{k=1}^{n-1} k^3 = \left(\sum_{k=1}^{n-1} k\right)^2$$

and let's show that it is true for n i.e

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$

We have that

$$\sum_{k=1}^{n} k^3 = \sum_{k=1}^{n-1} k^3 + n^3$$

Using the hypothesis that the equality is true for $n-1$, we can write

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n-1} k\right)^2 + n^3$$

But we know that

$$\left(\sum_{k=1}^{n-1} k\right)^2 = \left(\frac{n(n-1)}{2}\right)^2$$

Thus

$$\begin{aligned}\sum_{k=1}^{n} k^3 &= \left(\frac{n(n-1)}{2}\right)^2 + n^3 \\ &= n^2\left(\frac{n^2-2n+1}{4} + n\right) \\ &= n^2\left(\frac{n^2-2n+1+4n}{4}\right) \\ &= n^2\left(\frac{n^2+2n+1}{4}\right) \\ &= \frac{n^2(n+1)^2}{4} \\ &= \left(\frac{n(n+1)}{2}\right)^2 \\ &= \left(\sum_{k=1}^{n} k\right)^2\end{aligned}$$

Problem 1.3. ***Solution***
One example of such function is

$$f(x) = \begin{cases} (\sqrt{2})^{-\frac{1}{x}} & x \in (0, 0.5] \\ 1 - (\sqrt{2})^{-\frac{1}{1-x}} & x \in (0.5, 1) \end{cases}$$

The function f is strictly increasing and

$$\lim_{x\to 0} f(x) = 0 \quad \lim_{x\to 1} f(x) = 1$$

Thus

$$\inf_{x\in(0,1)} f(x) = 0 \qquad \sup_{x\in(0,1)} f(x) = 1$$

But $f(\cdot)$ does not have a maximum or a minimum in $(0,1)$.
A plot of this function is shown is Figure 1.1

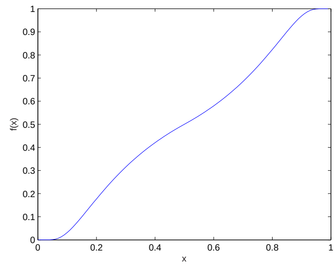

Figure 1.1. Plot of the function defined in exercise 3

Problem 1.4. ***Solution***
To compute the integral, we use a small trick.

$$\begin{aligned}\int_0^1 \frac{x+1}{x+2}dx &= \int_0^1 \frac{x+1+1-1}{x+2}dx \\ &= \int_0^1 (1+\frac{1}{x+2})dx \\ &= x]_0^1 - \int_0^1 \frac{1}{x+2}dx \\ &= 1 - [\log(x+2)]_0^1 \\ &= 1 - \log(3) + \log(2)\end{aligned}$$

Problem 1.5. ***Solution***

1. $0 \in (0,1)$, *False*
2. $0 \subset (-1,3)$, *True*
3. $(0,1) \cup (1,2) = (0,2)$, *False*
4. *The set of integers is uncountable, False*

Problem 1.6. ***Solution***
To compute the integral, we use integration by parts.
Consider $u = x^2$ *and* $v' = e^{-x}$*, we can rewrite the integral as (taking* $u' = 2x$ *and* $v = -e^{-x}$*):*

$$\begin{aligned}\int_0^\infty x^2e^{-x}dx &= \left[-x^2e^{-x}\right]_0^\infty + 2\int_0^\infty xe^{-x}dx && (1.1)\\ &= 0 + 2\int_0^\infty xe^{-x}dx && (1.2)\\ &= 2\left[-xe^{-x}\right]_0^\infty + 2\int_0^\infty e^{-x}dx && (1.3)\\ &= 0 + 2\left[-e^{-x}\right]_0^\infty && (1.4)\\ &= 2 && (1.5)\end{aligned}$$

where in equation 1.2 the first term vanishes as $x = 0$ and $x \to \infty$. In equation 1.3 we apply again the integration by parts with $u = x$ and $v' = e^{-x}$. The first term of the next equation vanishes and we get the result.

Problem 1.7. ***Solution***
We first write the expression for

$$B\Delta C = (B \cup C) - (B \cap C) = [0,4) - (2,3) = [0,2] \cup [3,4)$$

Now

$$A - (B\Delta C) = (1,5) - [0,2] \cup [3,4) = (2,3) \cup [4,5)$$

Problem 1.8. ***Solution***
To compute this double sum, we rewrite it in the following form:

	$n=0$	$n=1$	$n=2$	$\dots$	$n=N$
$m=0$	1	$1/2$	$1/3$	$\dots$	$1/(N+1)$
$m=1$		$1/2$	$1/3$	$\dots$	$1/(N+1)$
$m=2$			$1/3$	$\dots$	$1/(N+1)$
$\vdots$					$\vdots$
$m=N$					$1/(N+1)$
total $=$	1	$+2*1/2$	$+3*1/3$	$+\dots$	$+N*1/(N+1)$
$=$	1	$+1$	$+1$	$\dots$	1

Where in the last two rows we sum over all previous rows and columns.
This gives

$$\sum_{m=0}^{N}\sum_{n=m}^{N} = N+1$$

Problem 1.9. ***Solution***

$$\begin{aligned}\min\{A\} = \inf\{A\} &= 3\\ \max\{A\}: &\ \textit{does not exist}\\ \sup\{A\} &= 4.7\end{aligned}$$

Problem 1.10. ***Solution***
To show that $\inf(A) = -\sup(B)$*, we use the definitions of* sup *and* inf*. Let x be $x = \inf(A)$.*

$$\begin{aligned}x = \inf(A) &\Leftrightarrow \forall\epsilon, \exists y \in A, s.t, x \le y < x + \epsilon\\ &\Leftrightarrow \forall\epsilon, \exists y \in A, s.t, -x \ge -y > -x - \epsilon\end{aligned}$$

But if $y \in A$ then $-y = z \in B$ and the last equivalence can be written as

$$x = \inf(A) \Leftrightarrow \forall\epsilon, \exists z \in B, s.t, -x \ge z > -x - \epsilon$$

which implies that $-x = \sup(B)$.

Problem 1.11. ***Solution***
If $|a| < 1$ then we can write

$$\begin{aligned}\sum_{n=0}^{\infty} na^n = a\sum_{n=1}^{\infty} na^{n-1} &= a\left(\sum_{n=0}^{\infty} a^n\right)' \\ &= a\left(\frac{1}{1-a}\right)' \\ &= a\frac{1}{(1-a)^2}\end{aligned}$$

Similarly

$$\begin{aligned}\sum_{n=0}^{\infty} n^2a^n = a\sum_{n=0}^{\infty} n^2a^{n-1} &= a\left(\sum_{n=0}^{\infty} na^n\right)' \\ &= a\left(\frac{a}{(1-a)^2}\right)' \\ &= a\frac{1-a+2a}{(1-a)^3} \\ &= \frac{a(a+1)}{(1-a)^3}\end{aligned}$$

Problem 1.12. ***Solution***
We will first pick 3 red cars from the 26 red cards and 2 black cards from the 26 blacks, then we will mix them.
There are $\binom{26}{3}$ ways to select 3 red cards form the a deck and $\binom{26}{3}$ ways to select 2 black cards from the 26 black cards.
Once 5 cards have been selected, there are $\binom{5}{3}$ ways to mix them together.
Finally we have $\binom{26}{3}\binom{26}{2}\binom{5}{3}$ ways to select 5 cards with 3 red cards from a deck of 52 cards.

Problem 1.13. ***Solution***
We will show that $\sup(A)$ exists by explicitly computing it.
Define

$$B = \{x | x \geq y, \forall y \in A \quad \text{and} \quad x \leq b\}$$

Note that this set is not empty because $b \in B$. Furthermore, B is a closed set, so it admits a minimum which is equal to its inf (call it b_0). Now we want to show that $\sup(A) = b_0$. By definition we have that $b_0 \geq y$ for all $y \in A$, and $b_0 \leq b$. Since b_0 is defined as the $\inf(B)$, we have that for all $\epsilon > 0$, $b_0 - \epsilon \notin B$. But since $b_0 - \epsilon \leq b$, the only way for that to be possible is $b_0 - \epsilon < y$ for some $y \in A$. Thus

$$\forall \epsilon > 0, \exists y \in A, s.t. \quad b_0 - \epsilon < y \leq b_0$$

which means that $b_0 = \sup(A)$

Problem 1.14. ***Solution***
To derive an expression for the sum, we use the following trick:

$$\sum_{n=0}^{N} a^n = 1 + a + a^2 + \cdots + a_N$$
$$a\sum_{n=0}^{N} a^n = a + a^2 + \cdots + a^{N+1}$$

Now taking the difference of the two equations, and factorizing by $\sum_{n=0}^{N} a^n$ *we have*

$$(1-a)\sum_{n=0}^{N} a^n = 1 - a^{N+1} \Leftrightarrow \sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$$

Problem 1.15. ***Solution***
To show this, we will make use of Problem 13.
First let's define the set A *as*

$$A = \{x_n, n \geq 1\}$$

We know that A *is a set of real numbers and* a *is an upper bound for* A*. From Problem 13, we can deduce that* $x_s = \sup(A)$ *exists.*
Now let's show that

$$\lim_{n\to\infty} x_n = x_s$$

For that, first notice that x_n *is a non-decreasing sequence. Thus if* $x_{n_0} > x$*, then* $x_n > x$ *for all* $n \geq n_0$*. Since* $x_s = \sup(A)$*, we have*

$$\forall\epsilon > 0, \exists x_\epsilon \in A, s.t. x_s - \epsilon < x_\epsilon \leq x_s$$

But x_ϵ *is one element of the sequence* $\{x_n\}$*, and can be written* $x_\epsilon = x_{n_1}$ *for some* n_1*. Using the fact that the sequence is non-decreasing, we have*

$$x_s - \epsilon < x_{n_1} \leq x_n \leq x_s, \quad \forall n \geq n_1$$

Combining all we have:

$$\forall\epsilon > 0, \exists n_1 > 0, s.t., \forall n \geq n_1, \quad x_s - \epsilon < x_n \leq x_s$$

which means that $x_n \to x_s$ *as* $n \to \infty$*.*

Problem 1.16. ***Solution***
Let A_n *be the set of all sequences of characters of length* n*. We have* $|A_n| = 29^n$ *(all letters plus comma, dot, space...and whatever you want!). So* A_n *is countable.*
The set of English sentences of length n *is certainly included in* A_n*, hence the set of of all English sentences is included in* $\cup_n^{infty} A_n$*.*
But we know from the course note that if A_n *are countable for* $n \geq 1$*, then so is*

$$A = \cup_n^{\infty} A_n$$

which ends the proof.