Problem Set 2 — Due Feb, 1

Problem 2.1. *Pick 3 balls from an urn containing 15 balls (7 red balls, 5 blue balls, and 3 greens balls). Specify the probability space for this experiment.*

Solution:
The set of outcomes are all the triplet in the form

$$RRR, \quad RGB, \quad GBB, \quad GGG, \quad GRR, \quad BBR, etc \dots$$

The corresponding probabilities are

$$P[RRR] = \frac{7}{15}\frac{6}{14}\frac{5}{13}, \quad P[RGB] = \frac{7}{15}\frac{3}{14}\frac{5}{13}$$

$$P[GBB] = \frac{3}{15}\frac{5}{14}\frac{4}{13}, \quad P[GGG] = \frac{3}{15}\frac{2}{14}\frac{1}{13}$$

$$P[GRR] = \frac{3}{15}\frac{7}{14}\frac{6}{13}, \quad P[BBR] = \frac{5}{15}\frac{4}{14}\frac{7}{13}, etc \dots$$

Problem 2.2. *A part selected for testing is equally likely to have been produced on any one of six cutting tools.*

- *What is the sample space?*

- *What is the probability that the part is from tool 1?*

- *What is the probability that the part is from tool 1 or tool 3?*

- *What is the probability that the part is not from tool 5?*

Solution:

- The sample space can be written as

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

-

$$P[\text{part is from tool 1}] = \frac{1}{6}$$

- $$P[\text{part is from tool 1 or tool 3}] = \frac{2}{6}$$

- $$P[\text{part is not from tool 5}] = \frac{5}{6}$$

Problem 2.3. *Let A and B be two events. Use the axioms of probability to prove the following:*

1. $P(A \cap B) \geq P(A) + P(B) - 1$

2. *Show that the probability that one and only one of the events A or B occurs is $P(A) + P(B) - 2 \cdot P(A \cap B)$.*

Solution:

1. We have already proved in lecture and in the course notes that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

   Rearranging, we get

$$P(A \cap B) = P(A) + P(B) - P(A \cup B).$$

   Since $(A \cup B)$ is always a subset of Ω, the universal event, therefore, $P(A \cup B) \leq P(\Omega)$ and

$$P(A \cap B) \geq P(A) + P(B) - P(\Omega).$$

   Finally, by the normalization axiom, $P(\Omega) = 1$ and

$$P(A \cap B) \geq P(A) + P(B) - 1.$$

2. We begin by writing

$$
\begin{aligned}
P(A \text{ or } B, \text{ but not both}) &= P((A^c \cap B) \cup (A \cap B^c)) \\
&= P(A^c \cap B) + P(A \cap B^c),
\end{aligned}
$$

   where the last equality is from the additivity axiom. Next, we know that $B = (A^c \cap B) \cup (A \cap B)$ and $(A^c \cap B) \cap (A \cap B) = \emptyset$ so that we may apply the additivity axiom to get

$$P(B) = P(A^c \cap B) + P(A \cap B).$$

   With rearrangement, this becomes

$$P(A^c \cap B) = P(B) - P(A \cap B).$$

By symmetry, we also have

$$P(B^c \cap A) = P(A) - P(A \cap B).$$

So plugging in for $P(A^c \cap B)$ and $P(B^c \cap A)$, we get

$$
\begin{aligned}
P(A \text{ or } B, \text{ but not both}) &= P(B) - P(A \cap B) + P(A) - P(A \cap B) \\
&= P(A) + P(B) - 2P(A \cap B).
\end{aligned}
$$

Problem 2.4. *Measurements of the time needed to complete a chemical reaction might me modeled with the sample space $S = R^+$, the set of positive real numbers. Let*

$$E_1 = \{x | 1 \le x \le 10\} \quad \text{and} \quad E_2 = \{x | 3 \le x \le 118\}$$

Write the expressions for
$$E_1 \cup E_2, \quad E_1 \cap E_2, \quad E_1 \Delta E_2$$

Solution:

$$
\begin{aligned}
E_1 \cup E_2 &= \{x | 1 \le x \le 118\} \\
E_1 \cap E_2 &= \{x | 3 \le x \le 10\} \\
E_1 \Delta E_2 &= \{x | 1 \le x < 3\} \cup \{x | 1 < x \le 3\}
\end{aligned}
$$

Problem 2.5. *Consider two events, X_1 and X_2. Prove the following identities:*

1. $P(X_1 \cap X_2) \le P(X_1)$

2. $P(X_1) \le P(X_1 \cup X_2)$

3. $P(X_1 \cup X_2) \le P(X_1) + P(X_2)$

Solution:

1. By the monotonicity of the probability measure, since $X_1 \cap X_2$ is a subset of X_1,

$$\alpha \in X_1 \cap X_2 \Rightarrow \alpha \in X_1 \Rightarrow P(X_1 \cap X_2) \le P(X_1)$$

2. Similar to the argument in part (a) above, since X_1 is a subset of $X_1 \cup X_2$,

$$\alpha \in X_1 \Rightarrow \alpha \in X_1 \cup X_2 \Rightarrow P(X_1) \le P(X_1 \cup X_2)$$

3. Let $A = X_1 \cap X_2^c, B = X_2 \cap X_1^c, C = X_1 \cap X_2$, Observing that A, B are disjoint by construction, we have

$$X_1 \cup X_2 = A \cup B \cup C$$

$$P(X_1 \cup X_2) = P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

Since A is a subset of X_1 and B is a subset of X_2,

$$P(X_1) + P(X_2) = P(A) + P(B) + 2 * P(C)$$

and the result follows.

Problem 2.6. *Twenty distinct cars park in the same parking lot everyday. Ten of these cars are US-made, while the other ten are foreign-made. This parking lot has exactly twenty spaces, and all are in a row, so the cars park side by side each day. The drivers have different schedules on any given day, however, so the position any car might take on a certain day is random.*

1. *In how many different ways can the cars line up?*

2. *What is the probability that on a given day, the cars will park in such a way that they alternate (e.g., US-made, foreign-made, US-made, foreign-made, etc)?*

Solution:

1. Since the cars are all distinct, there are 20! ways to line them all up.

2. To find the probability that the cars will be parked in such a way that they will be alternating: US made, foreign made, etc... we will count the number of "favorable" outcomes, and divide by the total number of outcomes which we found in part (a) above. We count in the following manner: first lay the US cars down. We can do this in 10! ways, since the cars are distinct. Now lay the foreign cars in-between the US cars. Again we can do this in 10! ways. Finally, we need to multiply by 2, since the sequence could begin either with a US car or with a foreign car. Thus we have a total of $2 \cdot 10! \cdot 10!$, and the final answer is

$$\frac{2 \cdot 10! \cdot 10!}{20!}.$$

Note that we could have solved the second part of the problem by neglecting the fact that the cars are distinct. Suppose that the foreign cars are indistinguishable, and also suppose that the US cars are indistinguishable. Again we count the number of "favorable" outcomes in the same way: lay the US cars down in one way. Then there are two ways to lay the foreign cars down since the sequence can begin with either a US or a foreign car. Thus there are two favorable outcomes, out of a possible $\frac{20!}{10! \cdot 10!}$, and the two methods yield the same answer.

Problem 2.7. *Bob has a peculiar pair of four-sided dice. When he rolls the dice, the probability of any particular outcome is proportional to the sum of the outcome of each die. All outcomes that result in a particular sum are equally likely.*

1. *What is the probability of the sum being even?*

2. *What is the probability of Bob rolling a 4 and a 1?*

Solution:

The easiest way to solve this problem is to make a table of some sort, similar to the one below.

Die 1	Die 2	Sum	P(Sum)
1	1	2	2p
1	2	3	3p
1	3	4	4p
1	4	5	5p
2	1	3	3p
2	2	4	4p
2	3	5	5p
2	4	6	6p
3	1	4	4p
3	2	5	5p
3	3	6	6p
3	4	7	7p
4	1	5	5p
4	2	6	6p
4	3	7	7p
4	4	8	8p
		Total	80p

$$
\begin{aligned}
P(\textit{All events}) &= 1 \\
&= 80p (\textit{Total from the table}) \\
p &= \frac{1}{80}
\end{aligned}
$$

1.

$$
\begin{aligned}
P(\textit{sum being even}) &= 2p + 4p + 4p + 6p + 4p + 6p + 6p + 8p \\
&= 40p \\
&= \boxed{1/2}
\end{aligned}
$$

2.

$$
\begin{aligned}
P(\text{rolling a 4 and a 1}) &= P(1,4) + P(4,1) \\
&= 5p + 5p \\
&= 10p \\
&= \boxed{1/8}
\end{aligned}
$$

Problem 2.8. *A baseball pitcher, Bill, has good control of his pitches. He always throws his pitches inside the "box" which we consider to be a 2 by 2 square. He throws the pitches uniformly over the square (i.e. the probability of a pitch falling within an area of the square is proportional to this area.) Let $(0,0)$ and $(2,2)$ be the coordinates of the lower-left corner and the upper-right corner of the square, respectively as shown below.*
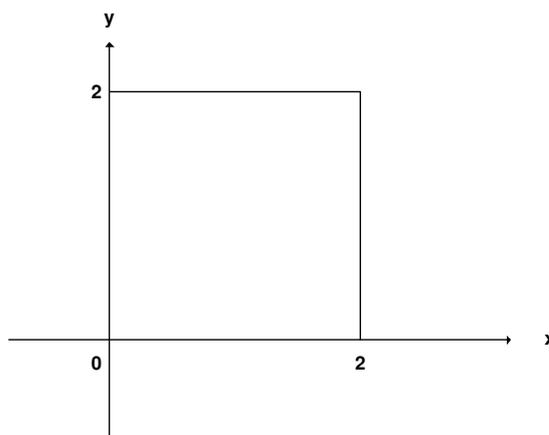
    Two groups A and B of fans are betting on where Bill's next pitch will fall. Among group A,

- *person 1 bets that the pitch is going to be in the left half part of the square, i.e. $0 \le x \le 1$.*

- *person 2 bets that it will be in one third of the square from the left, i.e. $0 \le x \le \frac{2}{3}$.*

- *and in general, person n makes the bet that the pitch will fall in the area $0 \le x \le 2/(n+1)$.*

1. *What is the probability that individual n from group A wins his bet?*

2. *What is the probability that individual n wins but not individual $n+1$?*

    Among group B, that fans bet in a similar fashion, but on the height of the pitch, i.e. individual n bets that the next pitch will fall in the area $0 \le y \le 2/(n+1)$.

  (c) What is the probability that individuals 1 through n of both groups win their bets?

(d) *When n goes to infinity, what is the probability that all fans of both groups win their bets? Note: Be precise in your derivation.*

Solution:

(a) The probability that individual n from group A wins his bet is equal to the ratio of the desired target area to the whole area of the square. Thus $P(\text{individual } n \text{ wins}) = \frac{\frac{4}{n+1}}{4} = \frac{1}{n+1}$.

(b) The probability that individual n from group A wins but not individual $n + 1 =$

$$\frac{\textit{desired target area of ind. } n - \textit{desired target area of ind. } n+1}{\textit{total area of square}} = \frac{\frac{4}{n+1} - \frac{4}{n+2}}{4} = \frac{1}{(n+1)(n+2)}.$$

(c) In order for individuals 1 through n of both groups to win, we must have $x \leq \frac{2}{n+1}$ and $y \leq \frac{2}{n+1}$. For instance, for individuals 1 and 2 from both groups to win their bets, Bill's pitch must land inside the square with vertices $(0,0)$, $(\frac{2}{3}, 0)$, $(\frac{2}{3}, \frac{2}{3})$, and $(0, \frac{2}{3})$.

So, the probability that individuals 1 through n win their bets is the ratio of the area of the square with vertices $(0,0)$, $(\frac{2}{n+1}, 0)$, $(\frac{2}{n+1}, \frac{2}{n+1})$, and $(0, \frac{2}{n+1})$ to the area of the whole square. This ratio equals $\frac{\frac{4}{(n+1)^2}}{4} = \frac{1}{(n+1)^2}$.

(d) Let A_i represent the event that individual i of group A wins his bet. Similarly, let B_i represent the event that individual i of group B wins his bet. For finitely many individuals, the probability that individuals 1 through n from both groups win their bets, $P(A_1 \cap A_2 ... \cap A_n \cap B_1 \cap B_2 ... \cap B_n)$, equals $\frac{1}{(n+1)^2}$.
As the number of individuals goes to infinity,

$$\lim_{n \to \infty} P(A_1 \cap A_2 ... \cap A_n \cap B_1 \cap B_2 ... \cap B_n) = \lim_{n \to \infty} \frac{1}{(n+1)^2} = 0$$

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