EE126: Probability and Random Processes **SP'07**

Problem Set 7 — Due March, 22

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Problem 7.1. *Let u and v be independent, standard normal random variables (i.e., u and v are independent Gaussian random variables with means of zero and variances of one). Let*

$$x = u + v$$

$$y = u - 2v.$$

1. *Do x and y have a bivariate normal distribution? Explain.*
2. *Provide a formula for E[$x|y$].*

Solution:

1. Recall that to write the joint p.d.f or a normal random vector, we need to invert its co-variance matrix. So for the p.d.f to be defined, the co-variance matrix must be invertible.
   Also we know that a jointly gaussian random vector is characterized by its mean and co-variance matrix. Hence in this exercise we just need to verify that the co-variance matrix of the vector $(x \quad y)^T$ is invertible.
   We have (taking into account the fact that the random variables u and v are zero mean)
   $$\Sigma = \begin{pmatrix} E[x^2] & E[xy] \\ E[xy] & E[y^2] \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$$
   and $det(\Sigma) = 16 \neq 0$.
   Thus $(x \quad y)^T$ has a joint p.d.f.

2. To compute E[$x|y$], we apply the formula given in the notes
   $$E[x|y] = \frac{\sigma_{xy}}{\sigma_y^2} y = \frac{-2}{5} y$$

Problem 7.2. *Let $X = (X_1, X_2, X_3)$ be jointly Gaussian with joint pdf*

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{e^{-(x_1^2 + x_2^2 - \sqrt{2}x_1x_2 + \frac{1}{2}x_3^2)}}{2\pi\sqrt{\pi}}$$

Find a transformation A such that $Y = AX$ consists of independent Gaussian random variables.

Solution:

In class we have seen that any jointly gaussian random vector X can be written as $X = BY$ where Y has i.i.d standard normal components, and $BB^T = \Sigma_X$. A formal way to compute B is to use the eigenvalue decomposition of the matrix $\Sigma_X = UDU^T$ where U is an orthogonal matrix and D is a diagonal matrix with non-negative entries in the diagonal. From this, B can be written $B = D^{\frac{1}{2}}U$. And, if Σ_X is invertible, A can be chosen as $A = B^{-1} = UD^{-\frac{1}{2}}$. If you are not familiar with eigenvalue decomposition, you can still compute B by solving the matrix equation $BB^T = \Sigma_X$.

First let's figure out what Σ_X is. For that, notice that

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \frac{1}{\sqrt{(2\pi)^2|\Sigma_X|}} e^{-\frac{1}{2}(x_1,x_2,x_3)\Sigma_X^{-1}(x_1,x_2,x_3)^T}$$

Developing the term in the exponent and identifying with the p.d.f given in the exercise, yield to

$$\Sigma_X = BB^T = \begin{pmatrix} 2 & -\sqrt{2} & 0 \\ -\sqrt{2} & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1.3066 & 0.5412 & 0 \\ 1.3066 & 0.5412 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1.3066 & 0.5412 & 0 \\ 1.3066 & 0.5412 & 0 \\ 0 & 0 & 1 \end{pmatrix}^T$$

which gives

$$A = B^{-1} = \begin{pmatrix} -0.3827 & 0.3827 & 0 \\ 0.9239 & 0.9239 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note:

Here, we have solved the exercise for Y having i.i.d standard normal components... but independent is enough and probably easier!

Problem 7.3. *A signal of amplitude $s = 2$ is transmitted from a satellite but is corrupted by noise, and the received signal is $Z = s + W$, where W is noise. When the weather is good, W is normal with zero mean and variance 1. When the weather is bad, W is normal with zero mean and variance 4. Good and bad weather are equally likely. In the absence of any weather information:*

1. *Calculate the PDF of Z.*
2. *Calculate the probability that Z is between 1 and 3.*

Solution:

1. Let G represent the event that the weather is good. We are given $P(G) = \frac{1}{2}$.

   To find the PDF of X, we first find the PDF of W, since $X = s + W = 2 + W$. We know that given good weather, $W \sim N(0,1)$. We also know that given bad weather,

$W \sim N(0,4)$. To find the unconditional PDF of W, we use the density version of the total probability theorem.

$$\begin{aligned} f_W(w) &= P(G) \cdot f_{W|G}(w) + P(G^c) \cdot f_{W|G^c}(w) \\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} + \frac{1}{2} \cdot \frac{1}{2\sqrt{2\pi}} e^{-\frac{w^2}{2(4)}} \end{aligned}$$

We now perform a change of variables using $X = 2 + W$ to find the PDF of X:

$$f_X(x) = f_W(x-2) = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2}} + \frac{1}{2} \cdot \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-2)^2}{8}}.$$

2. In principle, one can use the PDF determined in part (a) to compute the desired probability as

$$\int_1^3 f_X(x)\, dx.$$

It is much easier, however, to translate the event $\{1 \leq X \leq 3\}$ to a statement about W and then to apply the total probability theorem.

$$P(1 \leq X \leq 3) = P(1 \leq 2 + W \leq 3) = P(-1 \leq W \leq 1)$$

We now use the total probability theorem.

$$P(-1 \leq W \leq 1) \;=\; P(G)\underbrace{P(-1 \leq W \leq 1 \mid G)}_{a} + P(G^c)\underbrace{P(-1 \leq W \leq 1 \mid G^c)}_{b}$$

Since conditional on either G or G^c the random variable W is Gaussian, the conditional probabilities a and b can be expressed using Φ ($\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{x^2}{2}}\, dx$). Conditional on G, we have $W \sim N(0,1)$ so

$$a = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1.$$

Try to show that $\Phi(-x) = 1 - \Phi(x)$!
Conditional on G^c, we have $W \sim N(0,4)$ so

$$b = \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{2}\right) = 2\Phi\left(\frac{1}{2}\right) - 1.$$

The final answer is thus

$$P(1 \leq X \leq 3) = \frac{1}{2}\left(2\Phi(1) - 1\right) + \frac{1}{2}\left(2\Phi\left(\frac{1}{2}\right) - 1\right).$$

Problem 7.4. *Suppose X, Y are independent gaussian random variables with the same variance. Show that $X - Y$ and $X + Y$ are independent.*

Solution:

First note that, in general, to show that two random variables U and V are independent, we need to show that

$$P(U \in (u_1, u_2), V \in (v_1, v_2)) = P(U \in (u_1, u_2))P(V \in (v_1, v_2)), \quad \forall u_1, u_2, v_1, v_2$$

which can be very hard sometimes.
And also in general un-correlation is weaker than independence.
However for Gaussian random variables, we know that independence is equivalent to un-correlation. So, to show that U and V are independent, it suffices to show that they are un-correlated:

$$E[(U - E(U))(V - E[V])] = 0$$

We will apply this to $U = X + Y$ and $V = X - Y$. First notice that $E[U] = E[X] + E[Y]$ and $E[V] = E[X] - E[Y]$. Thus

$$\begin{aligned} E[X + Y - (E[X] + E[Y])(X - Y - (E[X] - E[Y]))] &= E[(X - E[X])^2] - E[(Y - E[Y])^2] \\ &\quad +E[(X - E[X])(Y - E[Y])] \\ &\quad -E[(X - E[X])(Y - E[Y])] \\ &= E[(X - E[X])^2] - E[(Y - E[Y])^2] \\ &= 0 \end{aligned}$$

because X and Y have equal variance.
So $X - Y$ and $X + Y$ are uncorrelated; since they are jointly gaussian (because they are linear combinations of the same independent random variables X and Y), they are independent.

Problem 7.5. *Steve is trying to decide how to invest his wealth in the stock market. He decides to use a probabilistic model for the shares price changes. He believes that, at the end of the day, the change of price Z_i of a share of a particular company i is the sum of two components: X_i, due solely to the performance of the company, and the other Y due to investors' jitter.*

Assuming that Y is a normal random variable, zero-mean and with variance equal to 1, and independent of X_i. Find the PDF of Z_i under the following circumstances in part a) to c),

1. *X_1 is Gaussian with a mean of 1 dollar and variance equal to 4.*
2. *X_2 is equal to -1 dollars with probability 0.5, and 3 dollars with probability 0.5.*
3. *X_3 is uniformly distributed between -2.5 dollars and 4.5 dollars (No closed form expression is necessary.)*

4. *Being risk averse, Steve now decides to invest only in the first two companies. He uniformly chooses a portion V of his wealth to invest in company 1 (V is uniform between 0 and 1.) Assuming that a share of company 1 or 2 costs 100 dollars, what is the expected value of the relative increase/decrease of his wealth?*

Solution:

1. Because Z_1 is the sum of two independent Gaussian random variables, X_1 and Y, the PDF of Z_1 is also Gaussian. The mean and variance of Z_1 are equal to the sums of the expected values of X_1 and Y and the sums of the variances of X_1 and Y, respectively.

$$f_{Z_1}(z_1) = N(1, 5)$$

2. X_2 is a two-valued discrete random variable, so it is convenient to use the total probability theorem and condition the PDF of Z_2 on the outcome of X_2. Because linear transformations of Gaussian random variables are also Gaussian, we obtain:

$$f_{Z_2}(z_2) = \frac{1}{2}N(-1, 1) + \frac{1}{2}N(3, 1)$$

3. We can use convolution here to get the PDF of Z_3.

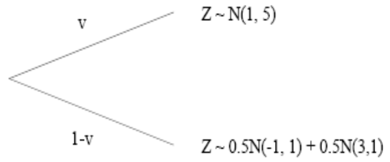
$$f_{Z_3}(z_3) = \int_{-\infty}^{\infty} N(0, 1) f_{X_3}(z_3 - y) dy$$

Using the fact that X_3 is uniform from -2.5 to 4.5, we can reduce this convolution to:

$$f_{Z_3}(z_3) = \int_{z_3-4.5}^{z_3+2.5} N(0, 1) \frac{1}{7} dy$$

A normal table is necessary to compute this integral for all values of z_3.

4. Given an experimental value $V = v$, we can draw the following tree:



$$E[Z] = E[E[Z \mid V]] = \int_0^1 E[Z \mid V = v] f_V(v) dv$$

$$E[Z \mid V = v] = v \cdot 1 + (1 - v)[-1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2}] = v + (1 - v) = 1$$

Plugging $E[Z \mid V = v]$ and $f_V(v) = 1$ into our first equation, we get

$$E[Z] = \int_0^1 1 \cdot 1dv = 1.$$

Since the problem asks for the relative change in wealth, we need to divide $E[Z]$ by 100. Thus the expected relative change in wealth is 1 percent.

Problem 7.6. *The Binary Phase-shift Keying (BPSK) and Quadrature Phase-shift Keying (QPSK) modulation schemes are shown in figure 7.1. We consider that in both cases, the symbols (S) are sent over an additive gaussian channel with zero mean and variance σ^2. Assuming that the symbols are equally likely, compute the average error probability for each scheme. Which one is better?*

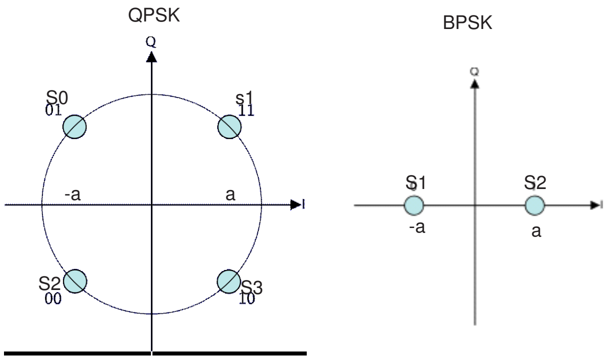


Figure 7.1. QPSK and BPSK modulations

Solution:(Hint)
Note that the comparison is not fair because the two schemes do not have the same rate (eggs and apples!). But let us compute the error probabilities and compare them.
In both cases, an error occurs when one symbol is sent and the receiver decides for a different one. Since the symbols are equally likely, the decoding (or decision) rule will be to decide in favor of the symbol S_i that maximizes the likelihood (assuming that Y is the received signal)

$$f_Y(y|S_i) = \frac{1}{\sqrt{2\pi \cdot 2}} e^{-\frac{(y-S_i)^T \cdot (y-S_i)}{2 \cdot 2}}$$

where $S_i \in \{-a, a\}$ for BPSK and $S_i \in \{(-a, -a), (-a, a), (a, -a), (a, a)\}$.
It is not hard to see that maximizing the likelihood is the same as minimizing $|y - S_i)|^2$ which turns out to be the Euclidean distance between the received point and the symbol S_i. Thus the decision rule is as follows (assuming that $Z \sim N(0, 2)$ is the channel noise):

- BPSK: decide for a if $X = S + Z \geq 0$ and decide for $-a$ otherwise. Error occurs if $S = a$ ($S = -a$) and $X = a + Z < 0 \Leftrightarrow Z < -a$ ($X = -a + Z \geq 0 \Leftrightarrow Z \geq a$). The corresponding conditional probabilities are $P(Z < -a) = \Phi(\sqrt{2}a)$ ($P(Z \geq a) = \Phi(\sqrt{2}a)$), thus the average error probability is $\Phi(\sqrt{2}a)$ for BPSK.

- QPSK: decide for (a, a) if X is in the first quartan, $(-a, a)$ if X is in the second quartan, etc...
  For QPSK (which can be modeled as 2 independent BPSK), let's assume that signal $S_1 = (a, a)$ was sent. Observe that error occurs if the received signal does not fall in the first quartan. By considering the probability of detecting any other signal, we can see that given S_1 was sent the error probability is equal to

  $$P_e^{qpsk} = \Phi(2a) + 2\Phi(\sqrt{2}a)$$

  which is the average error probability given the symmetry of the problem.
  For a large enough we have $P_e^{qpsk} \approx 2P_e^{bpsk}$

Problem 7.7. *When using a multiple access communication channel, a certain number of users N try to transmit information to a single receiver. If the real-valued random variable X_i represents the signal transmitted by user i, the received signal Y is*

$$Y = X_1 + X_2 + \cdots + X_N + Z,$$

where Z is an additive noise term that is independent of the transmitted signals and is assumed to be a zero-mean Gaussian random variable with variance σ_Z^2. We assume that the signals transmitted by different users are mutually independent and, furthermore, we assume that they are identically distributed, each Gaussian with mean μ and variance σ_X^2.

1. *If N is deterministically equal to 2, find the transform or the PDF of Y.*
2. *In most practical schemes, the number of users N is a random variable. Assume now that N is equally likely to be equal to $0, 1, \ldots, 10$.*
   (a) *Find the transform or the PDF of Y.*
   (b) *Find the mean and variance of Y.*
   (c) *Given that $N \geq 2$, find the transform or PDF of Y.*

Solution:

1. Here it is easier to find the PDF of Y. Since Y is the sum of independent Gaussian random variables, Y is Gaussian with mean 2μ and variance $2\sigma_X^2 + \sigma_Z^2$.

2. (a) The transform of N is

$$M_N(s) = \frac{1}{11}(1 + e^s + e^{2s} + \cdots + e^{10s}) = \frac{1}{11}\sum_{k=0}^{10} e^{ks}$$

Since Y is the sum of

- a random sum of Gaussian random variables
- an independent Gaussian random variable,

$$\begin{aligned}
M_Y(s) &= \left(M_N(s)|_{e^s = M_X(s)}\right) M_Z(s) = \left(\frac{1}{11}\sum_{k=0}^{10}(e^{s\mu + \frac{s^2\sigma_X^2}{2}})^k\right) e^{\frac{s^2\sigma_Z^2}{2}} \\
&= \left(\frac{1}{11}\sum_{k=0}^{10} e^{sk\mu + \frac{s^2 k\sigma_X^2}{2}}\right) e^{\frac{s^2\sigma_Z^2}{2}} \\
&= \frac{1}{11}\sum_{k=0}^{10} e^{sk\mu + \frac{s^2(k\sigma_X^2 + \sigma_Z^2)}{2}}
\end{aligned}$$

In general, this is *not* the transform of a Gaussian random variable.

(b) One can differentiate the transform to get the moments, but it is easier to use the laws of iterated expectation and conditional variance:

$$\begin{aligned}
EY &= EXEN + EZ = 5\mu \\
\mathrm{var}(Y) &= EN\mathrm{var}(X) + (EX^2)\mathrm{var}(N) + \mathrm{var}(Z) = 5\sigma_X^2 + 10\mu^2 + \sigma_Z^2
\end{aligned}$$

(c) Now, the new transform for N is

$$M_N(s) = \frac{1}{9}(e^{2s} + \cdots + e^{10s}) = \frac{1}{9}\sum_{k=2}^{10} e^{ks}$$

Therefore,

$$\begin{aligned}
M_Y(s) &= \left(M_N(s)|_{e^s = M_X(s)}\right) M_Z(s) = \left(\frac{1}{9}\sum_{k=2}^{10}(e^{s\mu + \frac{s^2\sigma_X^2}{2}})^k\right) e^{\frac{s^2\sigma_Z^2}{2}} \\
&= \left(\frac{1}{9}\sum_{k=2}^{10} e^{sk\mu + \frac{s^2 k\sigma_X^2}{2}}\right) e^{\frac{s^2\sigma_Z^2}{2}} \\
&= \frac{1}{9}\sum_{k=2}^{10} e^{sk\mu + \frac{s^2(k\sigma_X^2 + \sigma_Z^2)}{2}}
\end{aligned}$$