UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: PROBABLITY AND RANDOM PROCESSES

Problem Set 6
Spring 2007

Issued: Thursday, March 8, 2007 **Due:** Thursday, March 15, 2007

Problem 6.1

Let $\underline{V} = (X, Y)$ be a pair of zero mean jointly Gaussian random variables. Let K be the *covariance matrix* of $\underline{V}$ defined as

$$K = \begin{pmatrix} \mathbf{E}[X^2] & \mathbf{E}[XY] \\ \mathbf{E}[XY] & \mathbf{E}[Y^2] \end{pmatrix}.$$

1. Show that the joint PDF of X and Y can be written as

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{|K|}} e^{-\underline{v}^t K^{-1} \underline{v}} \quad ,$$

where $\underline{v}$ is the column vector $(x, y)^t$, $|K|$ denotes the determinant of the matrix K. and K^{-1} its inverse.

2. Let $Z = 2X + Y$ and $W = X - 2Y$.

   (a) Are Z and W jointly Gaussian?

   (b) Find the joint PDF of Z and W.

Problem 6.2

Let X_1, X_2 and X_3 be three independent, continuous random variables with the same distribution. Given that X_2 is smaller than X_3, what is the conditional probability that X_1 is smaller than X_2?

Problem 6.3

Let $X_1, X_2, \ldots, X_n$, where $n \geq 2$, be independent and identically-distributed continuous random variables with CDF $F(x)$ (and PDF $f(x)$). Define $Y = \max(X_1, \ldots, X_n)$, $Z = \min(X_1, \ldots, X_n)$ and $D = Y - Z$. Derive an expression for the joint CDF $F_{Y,Z}(y, z)$. Are Y and Z independent?

Problem 6.4

A wire connecting two locations serves as the transmission medium for ternary-valued messages; in other words, any message between locations is known to be one of three possible symbols, each occurring equally-likely. It is also known that any numerical value sent over this wire is subject to distortion; namely, if the value X is transmitted, the value Y received at the other end is described by $Y = X + N$ where the random variable N represents additive noise, assumed to be normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 4$.

1. Suppose the transmitter encodes the three types of messages with the values -1, 0 and 1. At the other end, the received message is decoded according to the following rules:

   - If $Y > \frac{1}{2}$, then conclude the value 1 was sent.
   - If $Y < -\frac{1}{2}$. then conclude the value -1 was sent.
   - If $-\frac{1}{2} \leq Y \leq \frac{1}{2}$, then conclude the value 0 was sent.

   Determine the probability of error for this encoding/decoding scheme. Reduce your calculations to a single numerical value.

2. In an effort to reduce the probability of error, the following modifications are made. The transmitter encodes the three types of messages with the values -2, 0 and 2 while the receiver's decoding rules are:

   - If $Y > 1$, then conclude the value 2 was sent.
   - If $Y < -1$. then conclude the value -2 was sent.
   - If $-1 \leq Y \leq 1$, then conclude the value 0 was sent.

   Repeat part (a) for this modified encoding/decoding scheme.

Problem 6.5

1. Let X be a zero-mean Gaussian random variable with variance σ^2. Derive an expression for $\mathbf{E}[X^n]$, the nth moment, that is valid for all $n \geq 0$. *Hint*: Relate the nth moment to the $(n-2)$th moment, then separately consider the cases of n even and n odd.

2. A *central Chi-Squared* random variable Y, with n degrees of freedom, is defined as

   $$Y = X_1^2 + X_2^2 + \cdots + X_n^2,$$

   where the X_i's are *independent and identically distributed* zero-mean Gaussian random variables with variance σ^2. Find the expectation and variance of Y.

Problem 6.6

An ee126 graduate opens a new casino in Las Vegas and decides to make the games more challenging from a probabilistic point of view. In a new version of roulette, each contestant spins the following kind of roulette wheel. The wheel has radius r and its perimeter is divided into 20 intervals, alternating red and black. The red intervals (along the perimeter) are twice the width of the black intervals (also along the perimeter). The red intervals all have the same length and the black intervals all have the same length. After the wheel is spun, the center of the ball is equally likely to settle in any position on the edge of the wheel; in other words, the angle of the final ball position (marked at the ball's center) along the wheel's perimeter is distributed uniformly between 0 and 2π radians.

(a) What is the probability that the center of the ball settles in a red interval?

(b) Let B denote the event that the center of the ball settles in a black interval. Find the conditional PDF $f_{Z|B}(z)$, where Z is the distance, along the perimeter of the roulette wheel, between the center of the ball and the edge of the interval immediately clockwise from the center of the ball?

(c) What is the unconditional PDF $f_Z(z)$?

Another attraction of the casino is the Gaussian slot machine. In this game, the machine produces independent identically distributed (IID) numbers $X_1, X_2, ...$ that have normal distribution $\mathcal{N}(0, \sigma^2)$. For every i, when the number X_i is positive, the player receives from the casino a sum of money equal to X_i. When X_i is negative, the player pays the casino a sum of money equal to $|X_i|$.

(d) What is the standard deviation of the net total gain of a player after n plays of the Gaussian slot machine?

(e) What is the probability that the absolute value of the net total gain after n plays is greater than $2\sqrt{n}\sigma$?