EE 102 # Homework #1 Solutions **Fall 2001**

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1. Review of integration

(a) We use integration by parts and get

$$\int_0^\pi t\cos(t)dt = \underbrace{[t\sin(t)]_0^\pi}_{0} - \int_0^\pi \sin(t)dt = \cos(\pi) - \cos(0) = \underline{-2}.$$

For the next integral we apply integration by parts.

$$\begin{aligned}\int_0^\pi t^2\sin(t)dt &= \left[-t^2\cos(t)\right]_0^\pi - 2\int_0^\pi t(-\cos(t))dt \\ &= \pi^2 + 2\left[[t\sin(t)]_0^\pi - \int_0^\pi \sin(t)dt\right] \\ &= \pi^2 + 2\left[\cos(\pi) - \cos(0)\right] = \underline{\pi^2 - 4}\end{aligned}$$

(b) With substitution $t - \tau = \sigma$, $d\tau = -d\sigma$ we get

$$A(t) = \int_0^t f(t-\tau)d\tau = \int_t^0 f(\sigma)(-d\sigma) = \int_0^t f(\sigma)d\sigma.$$

This can be rewritten with a factor of 1 inserted and partially integrated as

$$A(t) = \int_0^t 1\cdot f(\sigma)\ d\sigma = [\sigma f(\sigma)]_0^t - \int_0^t \sigma f'(\sigma)d\sigma.$$

Since the equation $A(t)$ is a function of t only (σ and τ are just integration variables and are exchangeable), we rewrite $A(t)$ as

$$A(t) = tf(t) - \int_0^t \tau f'(\tau)d\tau,$$

which is what we had to prove.

(c) We integrate over the following four regions separately, considering for the previous region in our results.

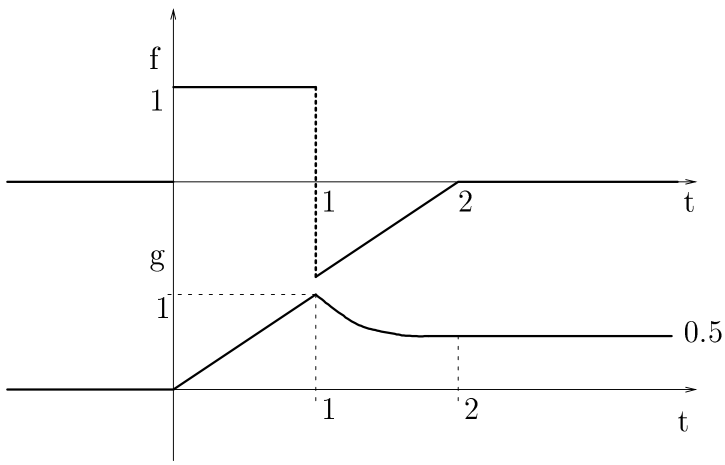
* $\mathbf{t < 0}$, $\underline{g(t) = 0}$.
* $\mathbf{0 \leq t \leq 1}$, $f(t) = 1$, $\underline{g(t)} = \int_0^t 1\ d\sigma \underline{= t}$ The previous result is always 0 so nothing has to be added.

* $\mathbf{1 \leq t \leq 2}$, $f(t) = t - 2$,

$$\underline{g(t)} = g(1) + \int_1^t (\sigma - 2)\ d\sigma = 1 + 0.5t^2 - 2t - 0.5 + 2 \underline{= 0.5t^2 - 2t + 2.5}.$$

  Be aware that the result of the previous region at the boundary $t = 1$, $g(1)$, has to be added.

* $\mathbf{2 \leq t}$, $f(t) = 0$, $\underline{g(t)} = g(2)\underline{= 0.5}$ The result is the previous result at $t = 2$ since nothing is added in region 4.



2. Review of complex numbers

   (a) Get real and imaginary parts

   (1) One full rotation (2π) of a vector (phasor) in the complex plane does not modify the vector and we get $e^{i\phi} = e^{i(\phi + k2\pi)}$ where k is an *integer* value. The problem can be seen as 6 full rotations plus a three quart rotation. The rotation is clockwise because the sign of the exponent is negative.

$$e^{-i\frac{27}{2}\pi} = e^{-i(6+\frac{3}{4})2\pi} = e^{-i\frac{3}{4}2\pi} = e^{-i3\frac{\pi}{2}} = \underline{i}$$

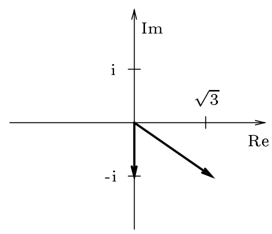
   (2) With $i = \sqrt{-1}$ and $i^2 = -1$ the problem can be written as

$$(i)^{i^6} = i^{(i^6)} = i^{((i^2)^3)} = i^{((-1)^3)} = i^{-1} = \frac{1}{i} = \frac{1 \cdot i}{i \cdot i} = \frac{i}{-1} = \underline{-i}.$$

   Another way to get the solution would use $i = e^{i\frac{\pi}{2}} = e^{-i3\frac{\pi}{2}}$ which gives the same result

$$i^{(i^6)} = i^{e^{i6\frac{\pi}{2}}} = i^{-1} = \underline{-i}.$$

(b) Get exponential form $|z|e^{i\phi}$



(1) From the figure it can be seen that $\phi = -\frac{\pi}{6}$ and the length of the vector $|z| = 2$.

$$\alpha = \sqrt{3} - i = \underline{2e^{-i\frac{\pi}{6}}}$$

(2) From the figure it can be seen that the vector has no real part. Its length is $|z| = 1$ and its phase is $\phi = -\frac{\pi}{2}$ which gives

$$\beta = -i = 1 \cdot e^{-i\frac{\pi}{2}} = \underline{e^{-i\frac{\pi}{2}}}.$$

(c) The complex conjugate of a number can be found in two ways. Either (i) negate its phase $\phi \to -\phi$, or (ii) negate its imaginary part Im $\to -$Im. We get

$$\frac{\alpha^3}{\overline{\beta}} = \frac{2^3 e^{-3i\frac{\pi}{6}}}{e^{i\frac{\pi}{2}}} = 8e^{-3i\frac{\pi}{6}} e^{-i\frac{\pi}{2}} = 8e^{-i\pi} = \underline{-8}.$$

(d) The equation can be written as $z^6 = 27$ and $z = 27^{\frac{1}{6}}$. In order to get all possible 6 results we use $27 = 27e^{i2k\pi}$ where k is any integer

$$z = \left[27e^{i2k\pi}\right]^{\frac{1}{6}} = \sqrt{3}e^{ik\frac{\pi}{3}}.$$

For $k = 0, 1, 2, 3, 4, 5$ we get the result as a set

$$\underline{z \in \left\{\sqrt{3}, \frac{\sqrt{3}}{2} + i\frac{3}{2}, -\frac{\sqrt{3}}{2} + i\frac{3}{2}, -\sqrt{3}, -\frac{\sqrt{3}}{2} - i\frac{3}{2}, \frac{\sqrt{3}}{2} - i\frac{3}{2}\right\}}.$$

3. Differential Equations

$$\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} - 2x(t)$$

The left and right hand side of the equation can be rewritten as

$$e^{-t}\frac{d}{dt}\left(e^t y(t)\right) = e^{2t}\frac{d}{dt}\left(e^{-2t}x(t)\right).$$

Multiplying by e^t and integrating from 0 to t yields

$$e^t y(t) - e^0 y(0) = \int_0^t e^{3\sigma} \frac{d}{d\sigma} \left(e^{-2\sigma} x(\sigma) \right) d\sigma.$$

Doing integration by parts for the right hand side using $u(t) = e^{3t}$ and $v(t) = e^{-2t} x(t)$ gives

$$e^t y(t) = e^t x(t) - 3 \int_0^t e^{3\sigma} e^{-2\sigma} x(\sigma) d\sigma.$$

The final result is

$$\underline{y(t) = x(t) - 3 \int_0^t e^{-(t-\sigma)} x(\sigma) d\sigma.}$$

4. System descriptions

   $x(t)$ is the input signal and the corresponding output of the system is defined as $y(t) = T[x(t)]$. For proof of system linearity the input signal is written as a linear combination $x(t) = \alpha x_1(t) + \beta x_2(t)$ and

   $$T[\alpha x_1(t) + \beta x_2(t)] = \alpha y_1(t) + \beta y_2(t)$$

   has to be true. For time invariance

   $$T[x(t-\tau)] = y(t-\tau)$$

   has to be true. Causality means that the output of the system $y(t)$ is not dependent on future values of the input $x(t)$.

   (a) $y(t) = x(t+1) - 3$
   Not linear: $T[\alpha x_1(t) + \beta x_2(t)] = \alpha x_1(t+1) + \beta x_2(t+1) - 3 \neq \alpha y_1(t) + \beta y_2(t) = \alpha(x_1(t+1) - 3) + \beta(x_2(t+1) - 3)$.
   Time invariant: $T[x(t-\tau)] = x(t-\tau+1) - 3 = y(t-\tau)$.
   Not causal: y at time t depends on value of x at time $t+1$, i.e. in the future.

   (b) $y(t) = e^t x(t)$
   Linear: $T[\alpha x_1(t) + \beta x_2(t)] = e^t(\alpha x_1(t) + \beta x_2(t)) = \alpha e^t x_1(t) + \beta e^t x_2(t) = \alpha y_1(t) + \beta y_2(t)$.
   Time variant: $T[x(t-\tau)] = e^t x(t-\tau) \neq y(t-\tau) = e^{(t-\tau)} x(t-\tau)$.
   Causal and memoryless: $y(t)$ depends only on x at current time.

(c) $y(t) = \int_t^\infty x(\sigma)d\sigma$

Linear: $T[\alpha x_1(t) + \beta x_2(t)] = \int_t^\infty (\alpha x_1(\sigma) + \beta x_2(\sigma))d\sigma = \alpha \int_t^\infty x_1(\sigma)d\sigma + \beta \int_t^\infty x_2(\sigma)d\sigma = \alpha y_1(t) + \beta y_2(t) = \alpha \int_t^\infty x_1(\sigma)d\sigma + \beta \int_t^\infty x_2(\sigma)d\sigma$.

Time invariant: $T[x(t-\tau)] = \int_t^\infty x(\sigma - \tau)d\sigma = \int_{(t-\tau)}^\infty x(\rho)d\rho = y(t-\tau)$.

Not causal: To find $y(t)$ you need values of x in the entire future (t, ∞).

(d) $y(t) = \begin{cases} x(t) & \text{if } x(t) > 0 \\ 0 & \text{else} \end{cases}$

Not linear. To make it simple we just consider one of the linearity conditions which is $T[\alpha x(t)] = \alpha y(t)$. If $x(t)$ is multiplied by a *negative* α then it can be seen that $x(t)$ switches its sign and $y(t)$ takes a totally different value. Example: $x(t) = 3$, which gives $y(t) = 3$. Now take $\alpha = -1$, $T[\alpha x(t)] = 0 \neq \alpha y(t) = -3$.

Time invariant: $T[x(t-\tau)] = \begin{cases} x(t-\tau) & \text{if } x(t-\tau) > 0 \\ 0 & \text{else} \end{cases} = y(t-\tau)$.

Causal and memoryless: $y(t)$ depends only on x at current time.