EE 102 # Homework #6 Solutions

Fall 2001 **Professor Paganini**

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1. (a) We found in HW #5 that $\omega_0 = \frac{\pi}{2}$ and the Fourier coefficients

$$F_n = \begin{cases} 0 & n \text{ odd}, n \neq \pm 1 \\ \frac{-2ni}{\pi(n^2-1)} & n \text{ even} \end{cases}$$

and

$$F_1 = F_{-1} = \frac{1}{2}$$

The sine-cosine Fourier series representation of $f(t)$ is

$$f(t) = a_0 + 2\sum_{n=1}^{\infty}(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$a_0 = F_0 = 0$$

$$a_n = \text{Re}[F_n] = \begin{cases} \frac{1}{2} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_n = -\text{Im}[F_n] = \begin{cases} 0 & n \text{ odd} \\ \frac{2n}{\pi(n^2-1)} & n \text{ even} \end{cases}$$

Hence the sine-cosine Fourier series representation of $f(t)$ is

$$\begin{aligned} f(t) &= \cos\left(\frac{\pi t}{2}\right) + \sum_{\substack{n = 2 \\ n \text{ even}}}^{\infty} \frac{4n}{\pi(n^2-1)} \sin\left(n(\frac{\pi}{2})t\right) \\ &= \cos\left(\frac{\pi t}{2}\right) + \sum_{k=1}^{\infty} \frac{8k}{\pi(4k^2-1)} \sin(k\pi t) \end{aligned}$$

(b) By Parseval's theorem,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |F_n|^2 &= \frac{1}{4}\int_0^2 \left(2\cos\left(\frac{\pi t}{2}\right)\right)^2 dt = \left(\frac{1}{2}\right)\left[\frac{1}{4}\int_0^4 \left(2\cos\left(\frac{\pi t}{2}\right)\right)^2 dt\right] \\ &= \left(\frac{1}{2}\right)\left(\frac{2^2}{2}\right) = 1 \end{aligned}$$

where we used the fact that $\frac{1}{T}\int_0^T \left(V_0 \cos\left(\frac{2\pi t}{T}\right)\right)^2 dt = \frac{V_0^2}{2}$

2. (a)

$$y(t) = \sum_{n=-\infty}^{\infty} Y_n e^{in\frac{2\pi}{T}t}$$

$$\begin{aligned} Y_n &= \frac{1}{T}\int_0^T y(t)e^{-in\frac{2\pi}{T}t}dt = \frac{1}{T}\int_0^{hT} V_0 e^{-in\frac{2\pi}{T}t}dt \\ &= \begin{cases} iV_0\left(\frac{e^{-i2\pi nh}-1}{2\pi n}\right) & n \neq 0 \\ hV_0 & n = 0 \end{cases} \end{aligned}$$

(b)

$$\begin{aligned} P(y) &= P_{AC}(y) + P_{DC}(y) = \frac{1}{T}\int_0^T |y(t)|^2 dt = \frac{1}{T}\int_0^{hT} V_0^2 dt \\ &= hV_0^2 \\ P_{DC}(y) &= |Y_0|^2 = h^2V_0^2 \end{aligned}$$

Hence

$$\frac{P_{AC}(y)}{P_{DC}(y)} = \frac{P(y) - P_{AC}(y)}{P_{DC}(y)} = \frac{hV_0^2 - h^2V_0^2}{h^2V_0^2} = \frac{1-h}{h}$$

(c) If

$$z(t) = \sum_{n=-\infty}^{\infty} Z_n e^{in\frac{2\pi}{T}t}$$

Then the DC term is

$$Z_0 = H(0).Y_0 = Y_0 = hV_0$$

(d)

$$Z_n = H(in\frac{2\pi}{T})Y_n = \frac{1}{1 + in\left(\frac{2\pi}{T}\right)(3T)}Y_n = \frac{1}{1+i(6n\pi)}Y_n$$

$$P_{AC}(z) = 2\sum_{n=1}^{\infty}|Z_n|^2 = 2\sum_{n=1}^{\infty}\frac{1}{1+36\pi^2n^2}|Y_n|^2$$

Since $\forall n$,

$$\frac{1}{1+36\pi^2n^2} \leq \frac{1}{1+36\pi^2} \leq \frac{1}{356}$$

$$P_{AC}(z) \leq \frac{1}{1+36\pi^2}\left(2\sum_{n=1}^{\infty}|Y_n|^2\right) \leq \frac{1}{356}P_{AC}(y)$$

Hence

$$\frac{P_{AC}(z)}{P_{DC}(z)} \leq \frac{1}{356}\frac{P_{AC}(y)}{P_{DC}(y)} = \frac{1}{356}\left(\frac{1-\frac{1}{2}}{\frac{1}{2}}\right) = \frac{1}{356}$$

We can get a tighter bound by noting that for $\tau = 3T$, and $n \neq 0$

$$Y_n = \begin{cases} 0 & n \text{ even} \\ \frac{V_0}{in\pi} & n \text{ odd} \end{cases}$$

Hence

$$\begin{aligned} P_{AC}(z) &= 2\sum_{n \text{ odd}, n>0} \frac{1}{1+36\pi^2 n^2}\frac{V_0^2}{n^2\pi^2} \\ &\leq 2\sum_{n \text{ odd}, n>0} \frac{1}{36\pi^2 n^2}\frac{V_0^2}{n^2\pi^2} \\ &= \frac{V_0^2}{18\pi^4}\sum_{n \text{ odd}, n>0}\frac{1}{n^4} = \frac{V_0^2}{1728}. \end{aligned}$$

using the fact that

$$\sum_{n \text{ odd}, n>0}\frac{1}{n^4} = \frac{\pi^4}{96}. \tag{1}$$

$$\implies \quad \frac{P_{AC}(z)}{P_{DC}(z)} \leq \frac{\frac{V_0^2}{1728}}{\frac{V_0^2}{4}} = \frac{1}{432}$$

Of course, we didn't expect you to know the sum (1), so the looser bound gets full credit. But we remark it is not difficult to show (1) by a Fourier argument.

3. One way is to use a trigonometry identity: $\sin(\theta_1)\sin(\theta_2) = \frac{\cos(\theta_1-\theta_2)-\cos(\theta_1+\theta_2)}{2}$, and get

$$f(t) = \sin(t)\sin(2t) = \frac{\cos(t-2t)-\cos(t+2t)}{2} = \frac{1}{2}\cos(t) - \frac{1}{2}\cos(3t)$$

Thus the sine-cosine Fourier series for $f(t)$ has just two terms. Using $\cos(t) = \frac{e^{it}+e^{-it}}{2}$ and $\cos(3t) = \frac{e^{i3t}+e^{-i3t}}{2}$, we get the exponential Fourier series to be

$$f(t) = -\frac{1}{4}e^{-i3t} + \frac{1}{4}e^{-it} + \frac{1}{4}e^{it} - \frac{1}{4}e^{i3t}. \tag{2}$$

But you don't need to remember these trigonometry formulas: you can instead use the Euler equations (these you must remember!)

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i}, \quad \sin(2t) = \frac{e^{i2t} - e^{-i2t}}{2i},$$

and multiply them to get directly (2), and from there the sine-cosine expansion.

If $f(t)$ is approximated using only the first harmonic term $\frac{1}{2}\cos(t)$, the error is the third harmonic term $\frac{1}{2}\cos(3t)$ Hence, the mean-square-error in the approximation would be

$$\overline{\epsilon_1^2} = \frac{(\frac{1}{2})^2}{2} = \frac{1}{8}$$

Alternatively,

$$\overline{\epsilon_1^2} = \sum_{|n|>1} |F_n|^2 = |F_{-3}|^2 + |F_3|^2 = \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{1}{8}.$$