EE 102

Homework #7 Solutions

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1. Fourier transforms

   a) $f_a(t) = u(t+1) - u(t-1)$ find $F_a(i\omega)$

   $$F_a(i\omega) = \int_{-\infty}^{\infty} [u(t+1) - u(t-1)]e^{-i\omega t}dt = \int_{-1}^{1} e^{-i\omega t}dt = \frac{1}{-i\omega}[e^{-i\omega t}]_{-1}^{1} = \frac{1}{-i\omega}[e^{-i\omega} - e^{i\omega}]$$

   $$= \frac{1}{-i\omega}(-2i)\sin(\omega) = 2\frac{\sin(\omega)}{\omega} = 2\text{sinc}(\omega)$$

   b) $f_b(t) = \cos(\pi t) + 1$ find $F_b(i\omega)$
   ¿From the lecture notes we get directly for $\cos(\pi t)$

   $$\pi\delta(\omega - \pi) + \pi\delta(\omega + \pi).$$

   For the 1 term we get

   $$2\pi\delta(\omega).$$

   The final result therefore is

   $$F_b(i\omega) = \pi\delta(\omega - \pi) + \pi\delta(\omega + \pi) + 2\pi\delta(\omega).$$

   c) We see that $f_c(t) = f_a(t) \cdot f_b(t)$. This problem can be solved in different ways. One way is the straightforward way by definition of the Fourier Transform. Another is to use the modulation property. Here it is solved using the fact that a multiplication in time-domain is a convolution in frequency domain $F_c(i\omega) = \frac{1}{2\pi}F_a(i\omega) * F_b(i\omega)$. Therefore

   $$F_c(i\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_a(\omega - \sigma)F_b(\sigma)d\sigma$$

   $$= \frac{1}{2}\int_{-\infty}^{\infty} F_a(\omega - \sigma)\delta(\sigma - \pi)d\sigma + \frac{1}{2}\int_{-\infty}^{\infty} F_a(\omega - \sigma)\delta(\sigma + \pi)d\sigma + \int_{-\infty}^{\infty} F_a(\omega - \sigma)\delta(\sigma)d\sigma$$

   $$= \frac{1}{2}F_a(\omega - \pi) + \frac{1}{2}F_a(\omega + \pi) + F_a(\omega)$$

   $$= \text{sinc}(\omega - \pi) + \text{sinc}(\omega + \pi) + 2\text{sinc}(\omega).$$

2. Inverse Fourier Transform from magnitude and phase. We write

$$F(\omega) = |F(\omega)|e^{i\theta_{\mathbb{F}}(\omega)}.$$

The phase can be written as

$$\theta_F(\omega) = \begin{cases} -\frac{\pi}{2}\omega + \frac{\pi}{2} & \text{if } \omega < 0 \\ -\frac{\pi}{2}\omega - \frac{\pi}{2} & \text{if } \omega > 0 \end{cases}.$$

The magnitude can be written as

$$|F(\omega)| = \begin{cases} 1 & \text{if } -2 \leq \omega \leq -1 \\ 1 & \text{if } 1 \leq \omega \leq 2 \\ 0 & \text{else} \end{cases}.$$

The inverse Fourier Transform is now

$$f(t) = \frac{1}{2\pi}\int_{-2}^{-1} 1e^{i(-\frac{\pi}{2}\omega+\frac{\pi}{2})}e^{i\omega t}d\omega + \frac{1}{2\pi}\int_{1}^{2} 1e^{i(-\frac{\pi}{2}\omega-\frac{\pi}{2})}e^{i\omega t}d\omega$$

$$= \frac{e^{i\frac{\pi}{2}}}{2\pi}\int_{-2}^{-1} e^{i(t-\frac{\pi}{2})\omega}d\omega + \frac{e^{-i\frac{\pi}{2}}}{2\pi}\int_{1}^{2} e^{i(t-\frac{\pi}{2})\omega}d\omega$$

$$= \frac{e^{i\frac{\pi}{2}}}{2\pi i(t-\frac{\pi}{2})}\left[e^{i(t-\frac{\pi}{2})\omega}\right]_{-2}^{-1} + \frac{e^{-i\frac{\pi}{2}}}{2\pi i(t-\frac{\pi}{2})}\left[e^{i(t-\frac{\pi}{2})\omega}\right]_{1}^{2}$$

$$= \frac{1}{2\pi(t-\frac{\pi}{2})}\left[e^{-i(t-\frac{\pi}{2})} - e^{-2i(t-\frac{\pi}{2})}\right] - \frac{1}{2\pi(t-\frac{\pi}{2})}\left[e^{2i(t-\frac{\pi}{2})} - e^{i(t-\frac{\pi}{2})}\right]$$

$$= \frac{1}{2\pi(t-\frac{\pi}{2})}\left[e^{-i(t-\frac{\pi}{2})} + e^{i(t-\frac{\pi}{2})} - e^{-2i(t-\frac{\pi}{2})} - e^{2i(t-\frac{\pi}{2})}\right]$$

$$= \frac{1}{2\pi(t-\frac{\pi}{2})}\left[2\cos(t-\frac{\pi}{2}) - 2\cos(2t-\pi)\right]$$

$$= \frac{1}{\pi(t-\frac{\pi}{2})}\left[\sin(t) + \cos(2t)\right].$$

3. a) Use the fact that $\mathcal{F}[\frac{B}{\pi}\text{sinc}Bt] = u(\omega+B) - u(\omega-B)$ which can be written as

$$\int_{-\infty}^{\infty} \frac{B}{\pi}\text{sinc}(Bt)e^{-i\omega t}dt = u(\omega+B) - u(\omega-B).$$

Letting $B = 1$ and multiplying the equation by π gives

$$F(i\omega) = \int_{-\infty}^{\infty} \text{sinc}(t)e^{-i\omega t}dt = \pi(u(\omega+1) - u(\omega-1)).$$

This result will be used in problem 3b). Evaluating the quation at $\omega = 0$ we get the form of the problem and the solution

$$\int_{-\infty}^{\infty} \text{sinc}(t)e^{-i0t}dt = \int_{-\infty}^{\infty} \text{sinc}(t)dt = \pi\left(u(1) - u(-1)\right) = \pi\left(1-0\right) = \pi.$$

b) Because $\frac{\sin(t)}{t}$ is real we can write

$$\left[\frac{\sin(t)}{t}\right]^2 = \frac{\sin(t)}{t}\overline{\frac{\sin(t)}{t}} = \left|\frac{\sin(t)}{t}\right|^2,$$

which reminds us of the Parseval Theorem

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(i\omega)|^2\,d\omega.$$

Combining above equations and $F(i\omega)$ from problem 3a) we can write

$$\int_{-\infty}^{\infty}\left[\frac{\sin(t)}{t}\right]^2 dt = \int_{-\infty}^{\infty}\left|\frac{\sin(t)}{t}\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\pi(u(\omega+1)-u(\omega-1))|^2\,d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1}|\pi|^2 d\omega = \frac{\pi}{2}[\omega]_{-1}^{1} = \frac{\pi}{2}[1+1] = \pi.$$

4. $f(i\omega) = \frac{1+i\omega}{1-i\omega}$

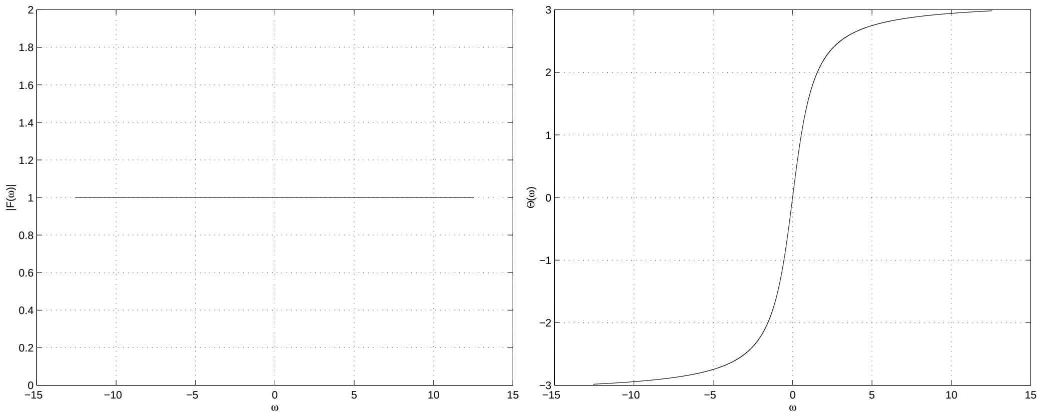
a) The magnitude is

$$|F(\omega)| = \frac{\sqrt{1^2+\omega^2}}{\sqrt{1^2+(-\omega)^2}} = 1.$$

This is an allpass filter. The phase is

$$\theta_F(\omega) = \tan^{-1}(\frac{\omega}{1}) - \tan^{-1}(\frac{-\omega}{1}) = \tan^{-1}(\omega) - \tan^{-1}(-\omega) = 2\tan^{-1}(\omega),$$

with $\theta_F(-\infty) = -\pi$ and $\theta_F(\infty) = \pi$.



b)

$$\mathcal{F}[g(t)] = \mathcal{F}[f(1-2t)]$$

$$\mathcal{F}[f(t)] = F(i\omega)$$

$$\mathcal{F}[f(t+1)] = e^{i\omega}F(i\omega)$$

$$\mathcal{F}[f(-2t+1)] = \frac{1}{|-2|}e^{-i\frac{\omega}{2}}F(-i\frac{\omega}{2}) = \frac{1}{2}e^{-i\frac{\omega}{2}}F(-i\frac{\omega}{2}) = \frac{1}{2}e^{-i\frac{\omega}{2}}\frac{2-i\omega}{2+i\omega}$$

c) We use the integrative property

$$\mathcal{F}\left[\int_{-\infty}^{t} f(\sigma)d\sigma\right] = \frac{F(i\omega)}{i\omega} + \pi F(0)\delta(\omega),$$

which yields

$$\mathcal{F}\left[\int_{-\infty}^{t} f(\sigma)d\sigma\right] = \frac{1+i\omega}{i\omega(1-i\omega)} + \pi\delta(\omega) = \frac{1+i\omega}{\omega^2+i\omega} + \pi\delta(\omega).$$