Stability Condition in Terms of the Pole Locations

- A causal LTI digital filter is BIBO stable if and only if its impulse response $h[n]$ is absolutely summable, i.e.,

$$S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- We now develop a stability condition in terms of the pole locations of the transfer function $H(z)$

Stability Condition in Terms of the Pole Locations

- The ROC of the z-transform $H(z)$ of the impulse response sequence $h[n]$ is defined by values of $|z| = r$ for which $h[n]r^{-n}$ is absolutely summable
- Thus, if the ROC includes the unit circle $|z| = 1$, then the digital filter is stable, and vice versa

Stability Condition in Terms of the Pole Locations

- In addition, for a stable and causal digital filter for which $h[n]$ is a right-sided sequence, the ROC will include the unit circle and entire z-plane including the point $z = \infty$

- An FIR digital filter with bounded impulse response is always stable

Stability Condition in Terms of the Pole Locations

- On the other hand, an IIR filter may be unstable if not designed properly
- In addition, an originally stable IIR filter characterized by infinite precision coefficients may become unstable when coefficients get quantized due to implementation

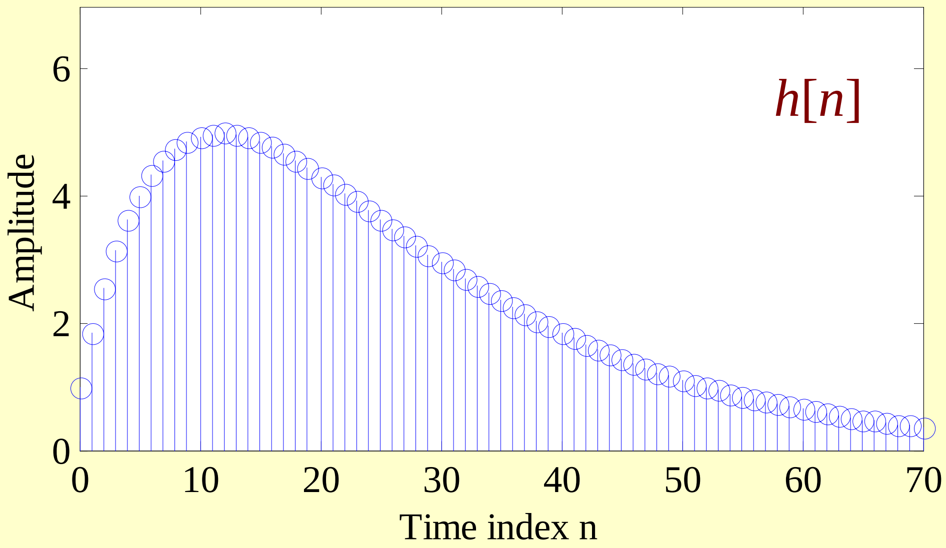
Stability Condition in Terms of the Pole Locations

- Example - Consider the causal IIR transfer function

$$H(z) = \frac{1}{1 - 1.845z^{-1} + 0.850586z^{-2}}$$

- The plot of the impulse response coefficients is shown on the next slide

Stability Condition in Terms of the Pole Locations



- As can be seen from the above plot, the impulse response coefficient $h[n]$ decays rapidly to zero value as n increases

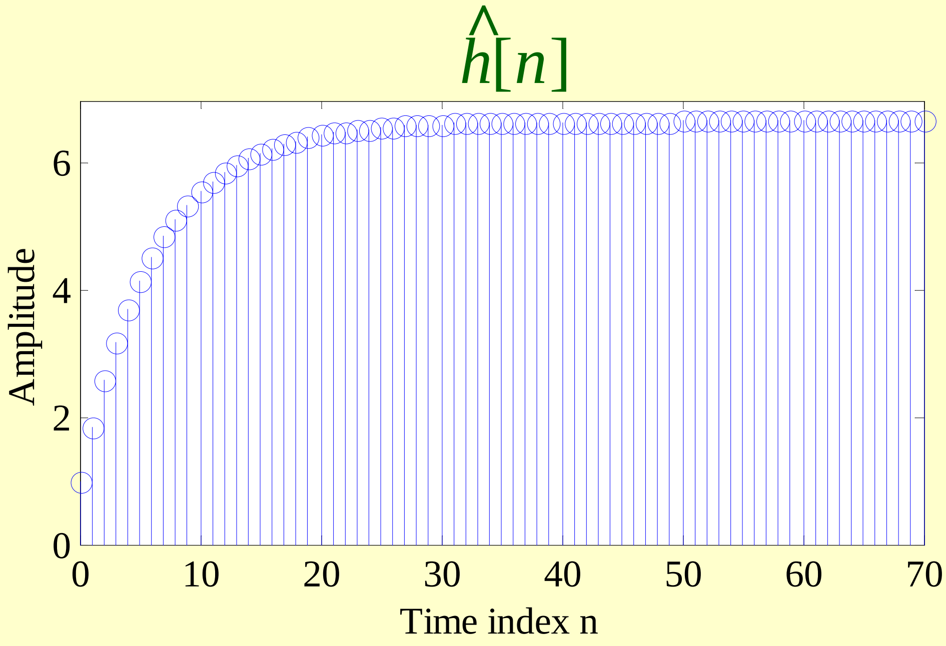
Stability Condition in Terms of the Pole Locations

- The absolute summability condition of $h[n]$ is satisfied
- Hence, $H(z)$ is a stable transfer function
- Now, consider the case when the transfer function coefficients are rounded to values with 2 digits after the decimal point:

$$\hat{H}(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

Stability Condition in Terms of the Pole Locations

- A plot of the impulse response of $\hat{h}[n]$ is shown below



Stability Condition in Terms of the Pole Locations

- In this case, the impulse response coefficient $\hat{h}[n]$ increases rapidly to a constant value as n increases
- Hence, the absolute summability condition of is violated
- Thus, $\hat{H}(z)$ is an unstable transfer function

Stability Condition in Terms of the Pole Locations

- The stability testing of a IIR transfer function is therefore an important problem
- In most cases it is difficult to compute the infinite sum

$$S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- For a causal IIR transfer function, the sum S can be computed approximately as

$$S_K = \sum_{n=0}^{K-1} |h[n]|$$

Stability Condition in Terms of the Pole Locations

- The partial sum is computed for increasing values of K until the difference between a series of consecutive values of S_K is smaller than some arbitrarily chosen small number, which is typically 10^{-6}
- For a transfer function of very high order this approach may not be satisfactory
- An alternate, easy-to-test, stability condition is developed next

Stability Condition in Terms of the Pole Locations

- Consider the causal IIR digital filter with a rational transfer function $H(z)$ given by

$$H(z) = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}$$

- Its impulse response $\{h[n]\}$ is a right-sided sequence
- The ROC of $H(z)$ is exterior to a circle going through the pole furthest from $z = 0$

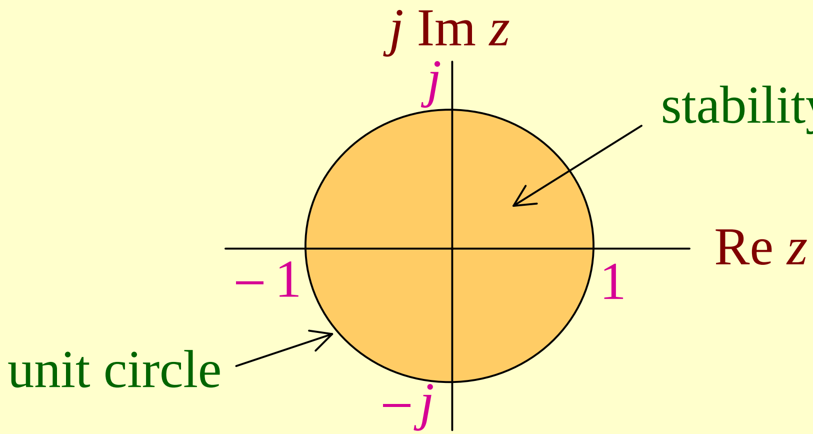
Stability Condition in Terms of the Pole Locations

- But stability requires that $\{h[n]\}$ be absolutely summable
- This in turn implies that the DTFT $H(e^{j\omega})$ of $\{h[n]\}$ exists
- Now, if the ROC of the z-transform $H(z)$ includes the unit circle, then

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

Stability Condition in Terms of the Pole Locations

- Conclusion: All poles of a causal stable transfer function $H(z)$ must be strictly inside the unit circle
- The stability region (shown shaded) in the z-plane is shown below



Stability Condition in Terms of the Pole Locations

- Example - The factored form of

$$H(z) = \frac{1}{1 - 0.845z^{-1} + 0.850586z^{-2}}$$

is

$$H(z) = \frac{1}{(1 - 0.902z^{-1})(1 - 0.943z^{-1})}$$

which has a real pole at $z = 0.902$ and a real pole at $z = 0.943$

- Since both poles are inside the unit circle, $H(z)$ is BIBO stable

Stability Condition in Terms of the Pole Locations

- Example - The factored form of

$$\hat{H}(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

is

$$\hat{H}(z) = \frac{1}{(1 - z^{-1})(1 - 0.85z^{-1})}$$

which has a real pole on the unit circle at $z = 1$ and the other pole inside the unit circle

- Since both poles are not inside the unit circle, $H(z)$ is unstable

Types of Transfer Functions

- The time-domain classification of an LTI digital transfer function sequence is based on the length of its impulse response:

  - Finite impulse response (FIR) transfer function

  - Infinite impulse response (IIR) transfer function

Types of Transfer Functions

- Several other classifications are also used
- In the case of digital transfer functions with frequency-selective frequency responses, one classification is based on the shape of the magnitude function $|H(e^{j\omega})|$ or the form of the phase function $\theta(\omega)$
- Based on this four types of ideal filters are usually defined

Ideal Filters

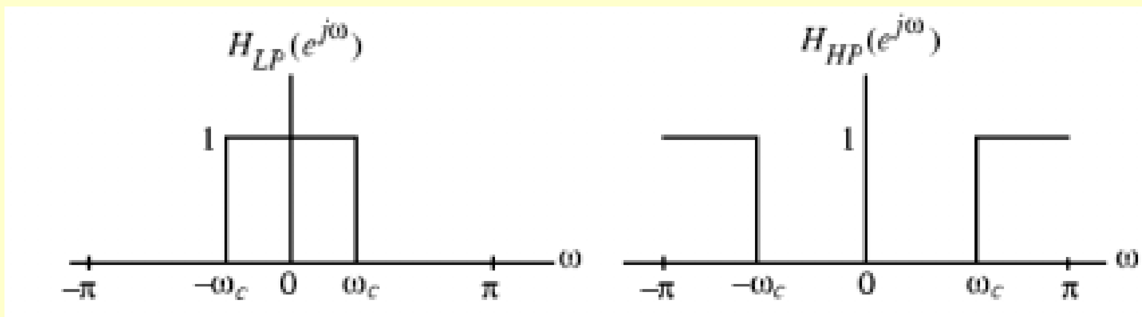
- A digital filter designed to pass signal components of certain frequencies without distortion should have a frequency response equal to one at these frequencies, and should have a frequency response equal to zero at all other frequencies

Ideal Filters

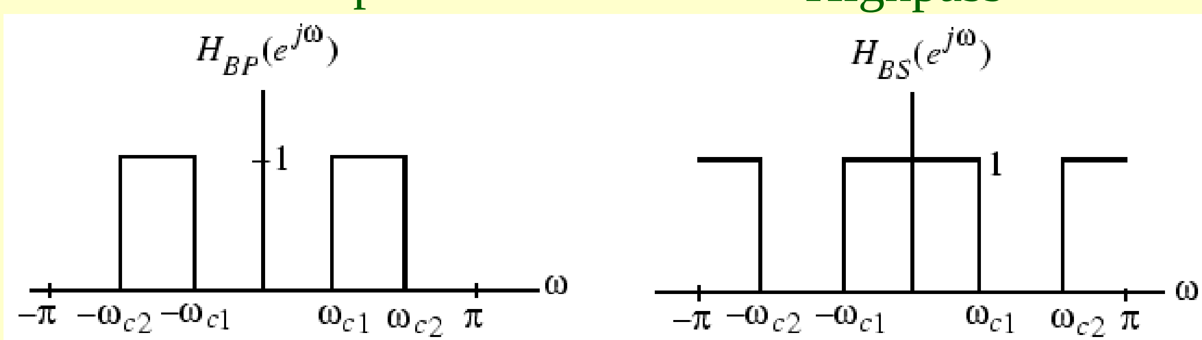
- The range of frequencies where the frequency response takes the value of one is called the **passband**
- The range of frequencies where the frequency response takes the value of zero is called the **stopband**

Ideal Filters

- Frequency responses of the four popular types of ideal digital filters with real impulse response coefficients are shown below:



Lowpass Highpass



Bandpass Bandstop

Ideal Filters

- Lowpass filter: Passband - $0 \le \omega \le \omega_c$
  Stopband - $\omega_c < \omega \le \pi$
- Highpass filter: Passband - $\omega_c \le \omega \le \pi$
  Stopband - $0 \le \omega < \omega_c$
- Bandpass filter: Passband - $\omega_{c1} \le \omega \le \omega_{c2}$
  Stopband - $0 \le \omega < \omega_{c1}$ and $\omega_{c2} < \omega \le \pi$
- Bandstop filter: Stopband - $\omega_{c1} < \omega < \omega_{c2}$
  Passband - $0 \le \omega \le \omega_{c1}$ and $\omega_{c2} \le \omega \le \pi$

Ideal Filters

- The frequencies ω_c , ω_{c1} , and ω_{c2} are called the **cutoff frequencies**
- An ideal filter has a magnitude response equal to one in the passband and zero in the stopband, and has a zero phase everywhere

Ideal Filters

- Earlier in the course we derived the inverse DTFT of the frequency response $H_{LP}(e^{j\omega})$ of the ideal lowpass filter:

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- We have also shown that the above impulse response is not absolutely summable, and hence, the corresponding transfer function is not BIBO stable

Ideal Filters

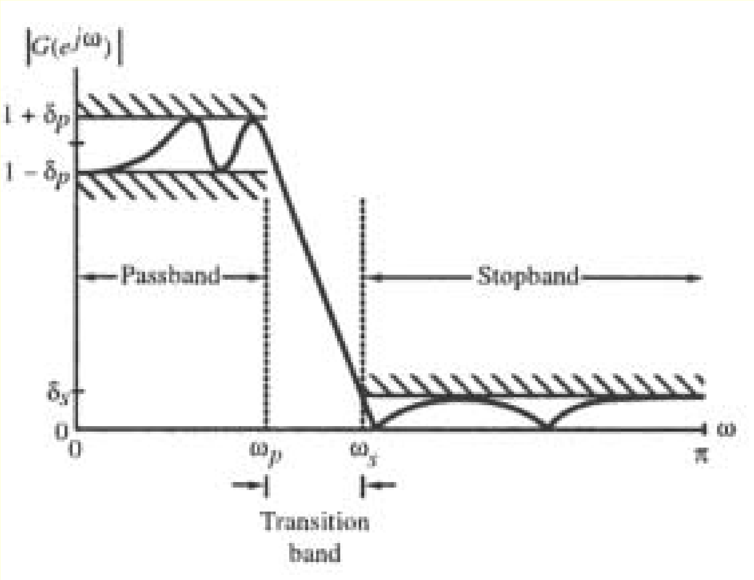
- Also, $h_{LP}[n]$ is not causal and is of doubly infinite length
- The remaining three ideal filters are also characterized by doubly infinite, noncausal impulse responses and are not absolutely summable
- Thus, the ideal filters with the ideal "brick wall" frequency responses cannot be realized with finite dimensional LTI filter

Ideal Filters

- To develop stable and realizable transfer functions, the ideal frequency response specifications are relaxed by including a **transition band** between the passband and the stopband
- This permits the magnitude response to decay slowly from its maximum value in the passband to the zero value in the stopband

Ideal Filters

- Moreover, the magnitude response is allowed to vary by a small amount both in the passband and the stopband
- Typical magnitude response specifications of a lowpass filter are shown below



Zero-Phase and Linear-Phase Transfer Functions

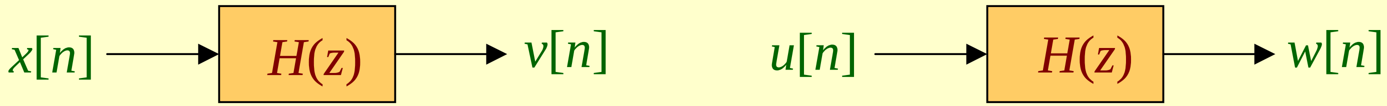
- A second classification of a transfer function is with respect to its phase characteristics
- In many applications, it is necessary that the digital filter designed does not distort the phase of the input signal components with frequencies in the passband

Zero-Phase and Linear-Phase Transfer Functions

- One way to avoid any phase distortion is to make the frequency response of the filter real and nonnegative, i.e., to design the filter with a **zero phase characteristic**
- However, it is possible to design a causal digital filter with a zero phase

Zero-Phase and Linear-Phase Transfer Functions

- For non-real-time processing of real-valued input signals of finite length, zero-phase filtering can be very simply implemented by relaxing the causality requirement
- One zero-phase filtering scheme is sketched below

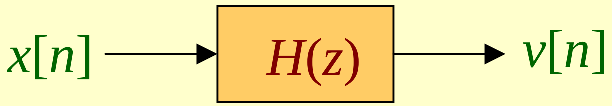


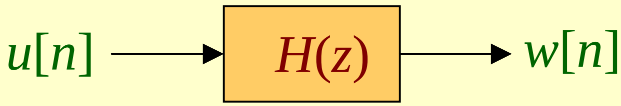
$$u[n] = v[-n], \qquad y[n] = w[-n]$$

Zero-Phase and Linear-Phase Transfer Functions

- It is easy to verify the above scheme in the frequency domain
- Let $X(e^{j\omega})$, $V(e^{j\omega})$, $U(e^{j\omega})$, $W(e^{j\omega})$, and $Y(e^{j\omega})$ denote the DTFTs of $x[n]$, $v[n]$, $u[n]$, $w[n]$, and $y[n]$, respectively
- From the figure shown earlier and making use of the symmetry relations we arrive at the relations between various DTFTs as given on the next slide

Zero-Phase and Linear-Phase Transfer Functions





$$u[n] = v[-n], \qquad y[n] = w[-n]$$

$$V(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \qquad W(e^{j\omega}) = H(e^{j\omega})U(e^{j\omega})$$

$$U(e^{j\omega}) = V^*(e^{j\omega}), \qquad Y(e^{j\omega}) = W^*(e^{j\omega})$$

- Combining the above equations we get

$$Y(e^{j\omega}) = W^*(e^{j\omega}) = H^*(e^{j\omega})U^*(e^{j\omega})$$

$$= H^*(e^{j\omega})V(e^{j\omega}) = H^*(e^{j\omega})\,H(e^{j\omega})\,X(e^{j\omega})$$

$$= \left|H(e^{j\omega})\right|^2 X(e^{j\omega})$$

Zero-Phase and Linear-Phase Transfer Functions

- The function `fftfilt` implements the above zero-phase filtering scheme
- In the case of a causal transfer function with a nonzero phase response, the phase distortion can be avoided by ensuring that the transfer function has a unity magnitude and a **linear-phase** characteristic in the frequency band of interest

Zero-Phase and Linear-Phase Transfer Functions

- The most general type of a filter with a linear phase has a frequency response given by

$$H(e^{j\omega}) = e^{-j\omega D}$$

which has a linear phase from $\omega = 0$ to $\omega = 2\pi$

- Note also $\left|H(e^{j\omega})\right| = 1$

$$\tau(\omega) = D$$

Zero-Phase and Linear-Phase Transfer Functions

- The output $y[n]$ of this filter to an input $x[n] = Ae^{j\omega n}$ is then given by

$$y[n] = Ae^{-j\omega D}e^{j\omega n} = Ae^{j\omega(n-D)}$$

- If $x_a(t)$ and $y_a(t)$ represent the continuous-time signals whose sampled versions, sampled at $t = nT$, are $x[n]$ and $y[n]$ given above, then the delay between $x_a(t)$ and $y_a(t)$ is precisely the group delay of amount D

Zero-Phase and Linear-Phase Transfer Functions

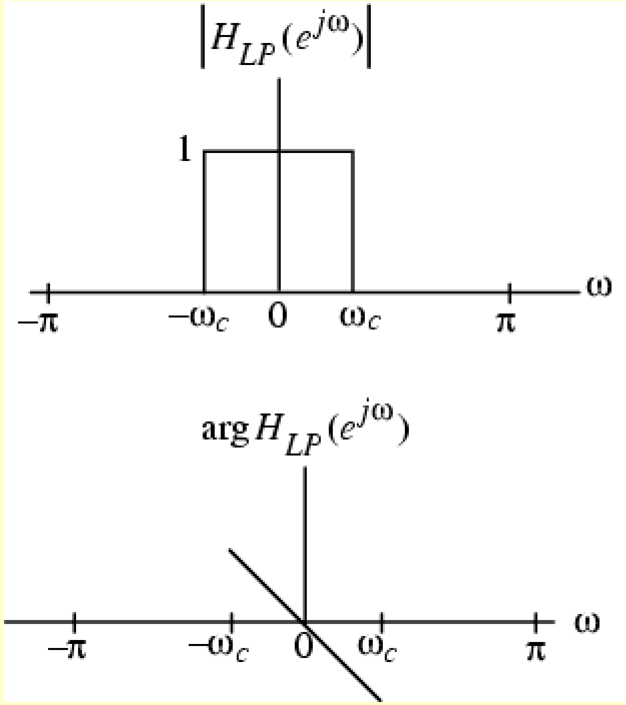
- If D is an integer, then $y[n]$ is identical to $x[n]$, but delayed by D samples
- If D is not an integer, $y[n]$, being delayed by a fractional part, is not identical to $x[n]$
- In the latter case, the waveform of the underlying continuous-time output is identical to the waveform of the underlying continuous-time input and delayed D units of time

Zero-Phase and Linear-Phase Transfer Functions

- If it is desired to pass input signal components in a certain frequency range undistorted in both magnitude and phase, then the transfer function should exhibit a unity magnitude response and a linear-phase response in the band of interest

Zero-Phase and Linear-Phase Transfer Functions

- Figure below shows the frequency response if a lowpass filter with a linear-phase characteristic in the passband



Zero-Phase and Linear-Phase Transfer Functions

- Since the signal components in the stopband are blocked, the phase response in the stopband can be of any shape
- Example - Determine the impulse response of an ideal lowpass filter with a linear phase response:

$$H_{LP}(e^{j\omega}) = \begin{cases} e^{-j\omega n_o}, & 0 < |\omega| < \omega_c \\ 0, & \omega_c \leq |\omega| \leq \pi \end{cases}$$

Zero-Phase and Linear-Phase Transfer Functions

- Applying the frequency-shifting property of the DTFT to the impulse response of an ideal zero-phase lowpass filter we arrive at

$$h_{LP}[n] = \frac{\sin \omega_c (n - n_o)}{\pi (n - n_o)}, \quad -\infty < n < \infty$$

- As before, the above filter is noncausal and of doubly infinite length, and hence, unrealizable

Zero-Phase and Linear-Phase Transfer Functions

- By truncating the impulse response to a finite number of terms, a realizable FIR approximation to the ideal lowpass filter can be developed
- The truncated approximation may or may not exhibit linear phase, depending on the value of n_o chosen

Zero-Phase and Linear-Phase Transfer Functions

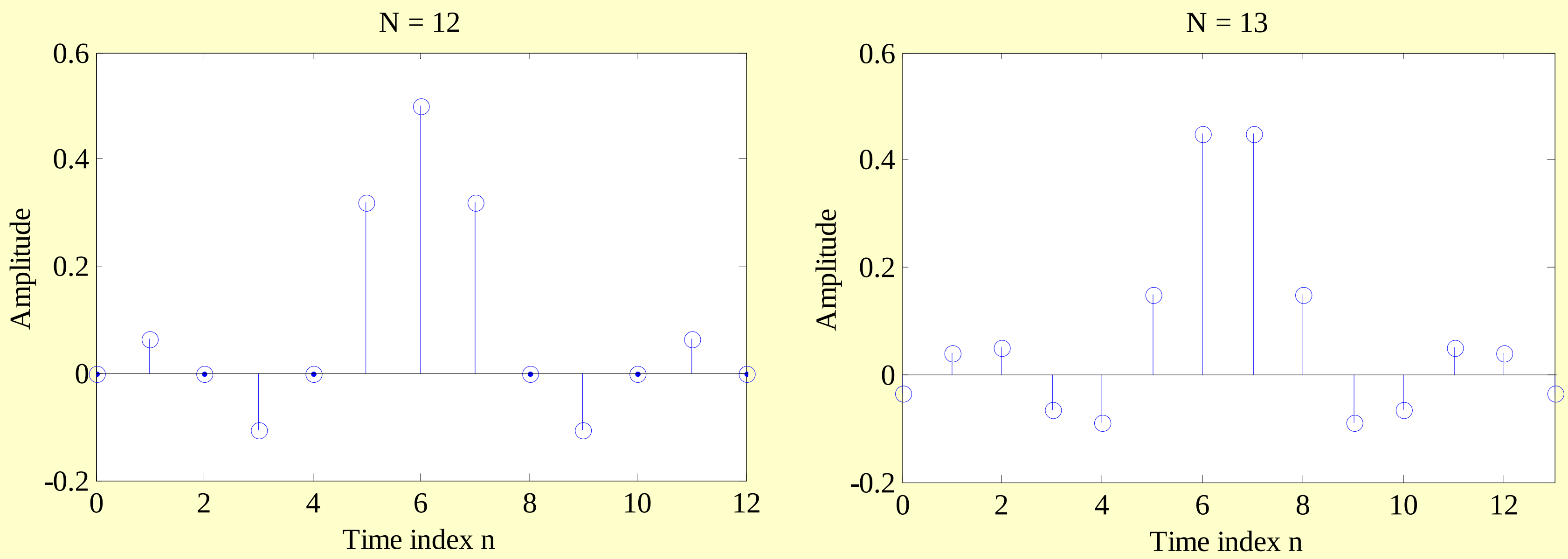
- If we choose $n_o = N/2$ with N a positive integer, the truncated and shifted approximation

$$\hat{h}_{LP}[n] = \frac{\sin \omega_c (n - N/2)}{\pi (n - N/2)}, \quad 0 \le n \le N$$

will be a length N+1 causal linear-phase FIR filter

Zero-Phase and Linear-Phase Transfer Functions

- Figure below shows the filter coefficients obtained using the function `sinc` for two different values of N



Zero-Phase and Linear-Phase Transfer Functions

- Because of the symmetry of the impulse response coefficients as indicated in the two figures, the frequency response of the truncated approximation can be expressed as:

$$\hat{H}_{LP}(e^{j\omega}) = \sum_{n=0}^{N} \hat{h}_{LP}[n] e^{-j\omega n} = e^{-j\omega N/2}\, \tilde{H}_{LP}(\omega)$$

where $\tilde{H}_{LP}(\omega)$, called the **zero-phase response** or **amplitude response**, is a real function of ω