# Filter Approximation Theory

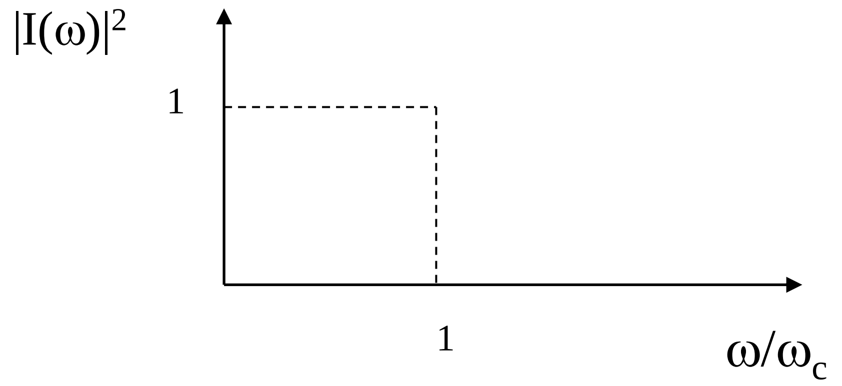
Butterworth, Chebyshev, and Elliptic Filters

# Approximation Polynomials

- Every physically realizable circuit has a transfer function that is a rational polynomial in s
- We want to determine classes of rational polynomials that approximate the “Ideal” low-pass filter response (high-pass band-pass and band-stop filters can be derived from a low pass design)
- Four well known approximations are discussed here:
  - Butterworth: Steven Butterworth,"On the Theory of Filter Amplifiers", Wireless Engineer (also called Experimental Wireless and the Radio Engineer), vol. 7, 1930, pp. 536-541
  - Chebyshev: Pafnuty Lvovich Chebyshev (1821-1894) - Russia
    Cyrillic alphabet - Spelled many ways Чебышёв
  - Elliptic Function: Wilhelm Cauer (1900-1945) - Germany
    U.S. patents 1,958,742 (1934), 1,989,545 (1935), 2,048,426 (1936)
  - Bessel: Friedrich Wilhelm Bessel, 1784 - 1846

# Definitions

- Let $|H(\omega)|^2$ be the approximation to the ideal low-pass filter response $|I(\omega)|^2$



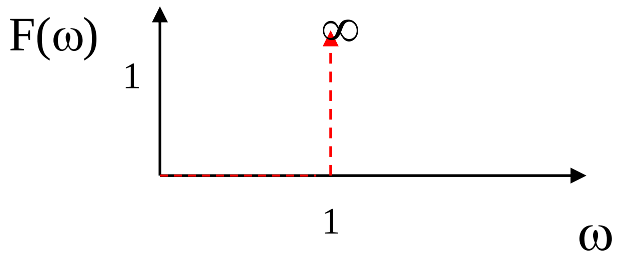

Where $\omega_c$ is the ideal filter cutoff frequency (it is normalized to one for convenience)

# Definitions - 2

- $|H(\omega)|^2$ can be written as

$$|H(\omega)|^2 = \frac{1}{1+\varepsilon^2 F^2(\omega)}$$

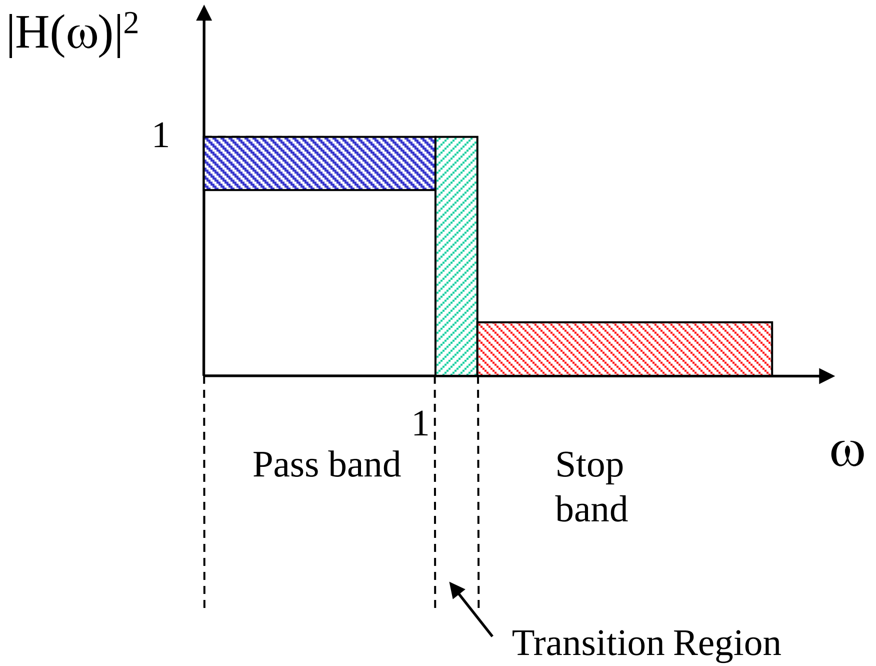
Where F(ω) is the "Characteristic Function" which attempts to approximate:




  - This cannot be done with a finite order polynomial
  - ε provides flexibility for the degree of error in the passband or stopband.

# Filter Specification

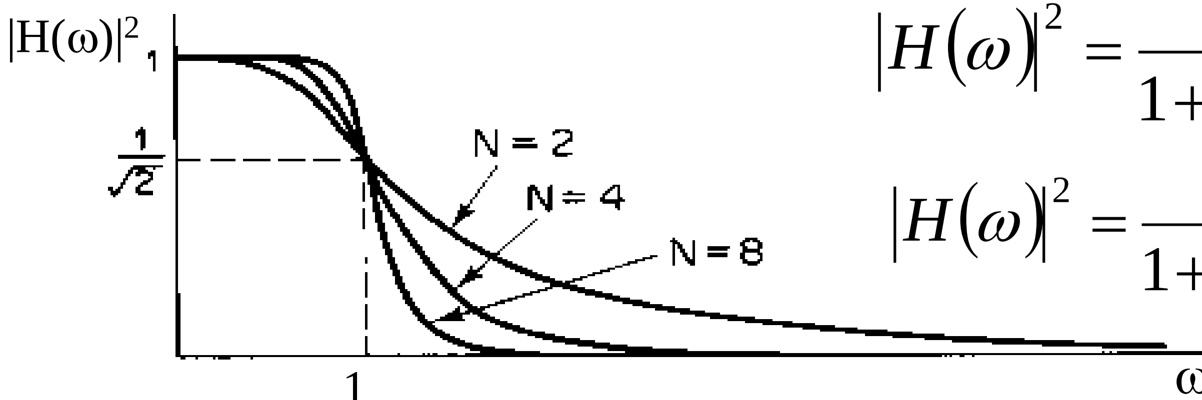
- $|H(\omega)|^2$ must stay within the shaded region




- Note that this is an incomplete specification. The phase response and transient response are also important and need to be appropriate for the filter application

# Butterworth

- $F(\omega) = \omega^n$ and $\varepsilon = 1$ and



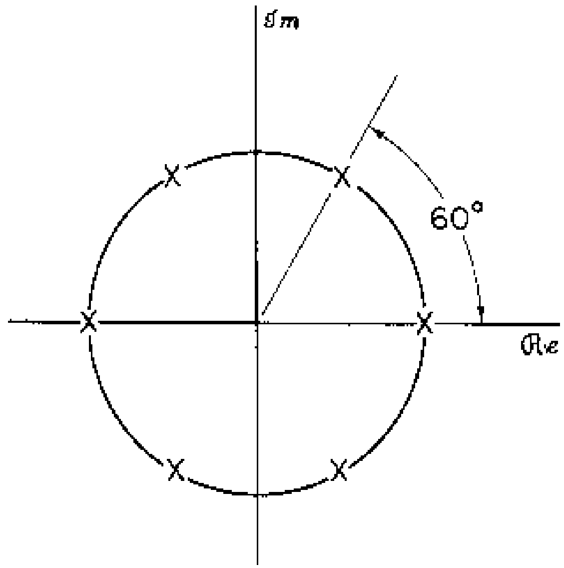

$$\left|H(\omega)\right|^2 = \frac{1}{1+\varepsilon^2 F^2(\omega)}$$

$$\left|H(\omega)\right|^2 = \frac{1}{1+\omega^{2n}}$$

- Characteristics
  - Smooth transfer function (no ripple)
  - Maximally flat and Linear phase (in the pass-band)
  - Slow cutoff ☹

# Butterworth Continued

- Pole locations in the s-plane at: $\left|H(\omega)\right|^2 = \dfrac{1}{1+\omega^{2n}}$
  $\omega^{2n} = -1$ or $\omega = (-1)^{(1/2n)}$

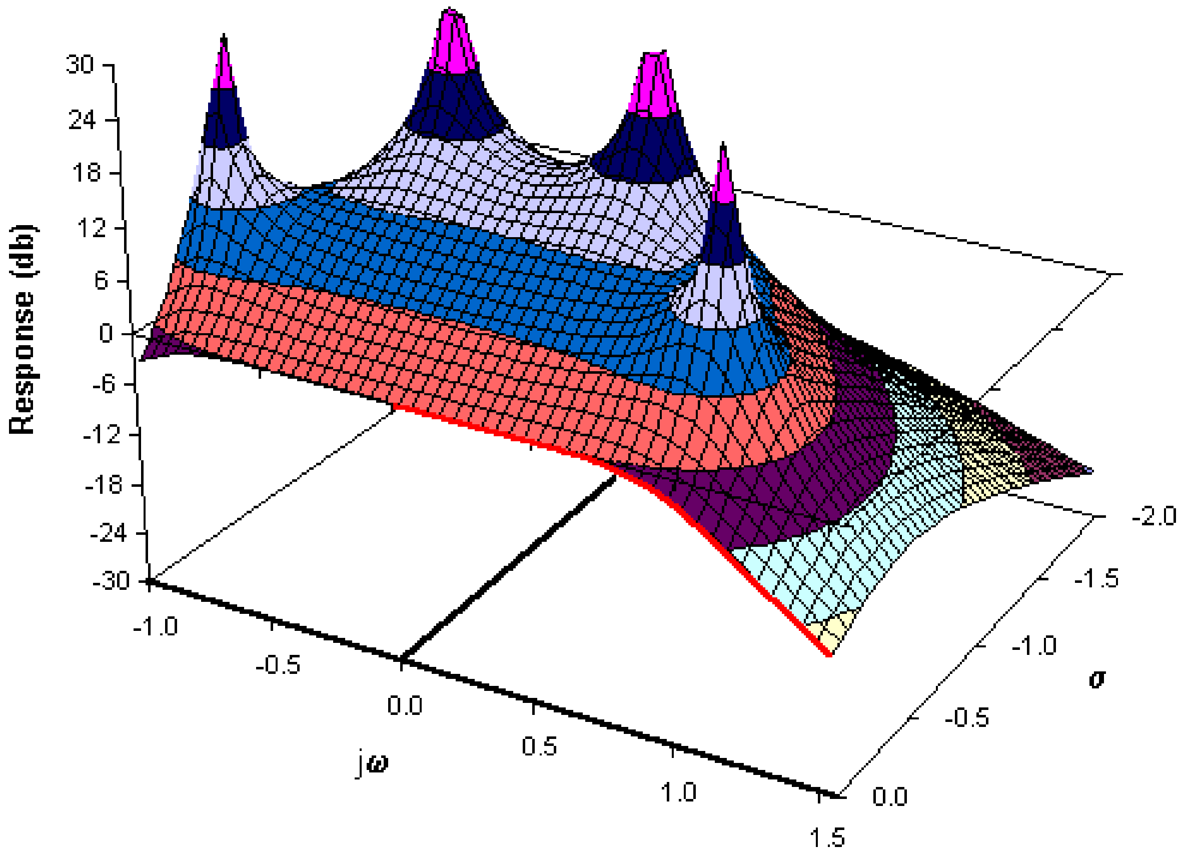


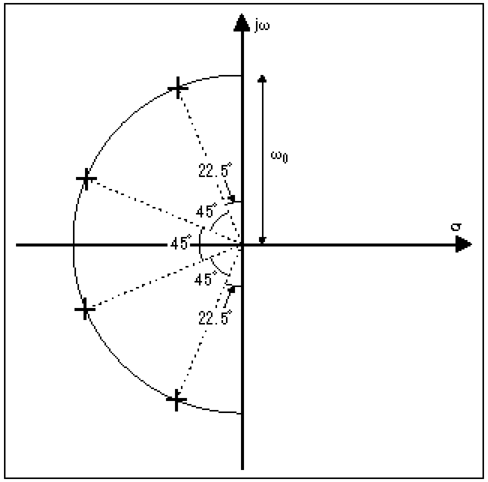

n = 3

  - Poles are equally spaced on the unit circle at $\theta = k\pi/2n$.
  - H(s) only uses the n poles in the left half plane for stability.
  - There are no zeros

# Butterworth Filter

## |H(s)| for n=4



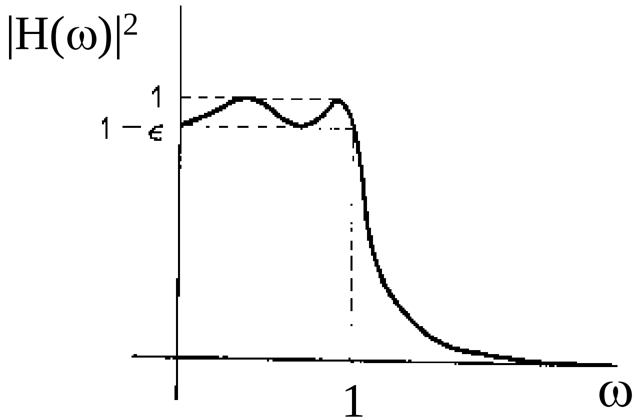



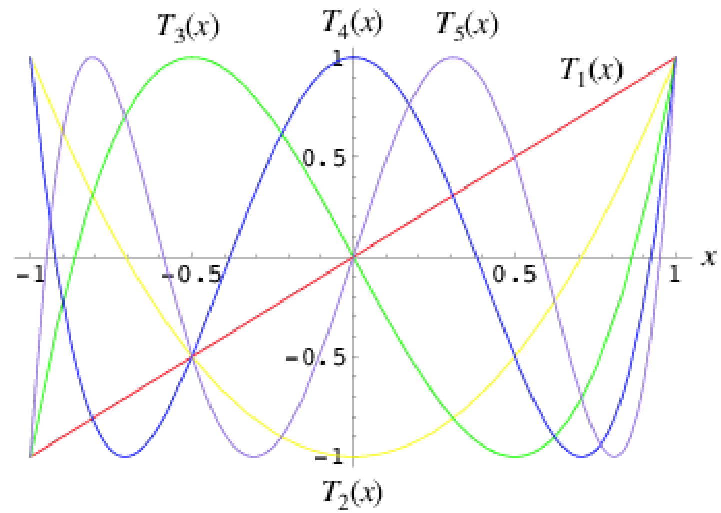

$$H(s) = 1/( s^4 + 2.6131s^3 + 3.4142s^2 + 2.6131s + 1)$$

# Chebyshev – Type 1

- $F(\omega) = T_n(\omega)$ so $$|H(\omega)|^2 = \frac{1}{1+\varepsilon^2 T_n^2(\omega)}$$

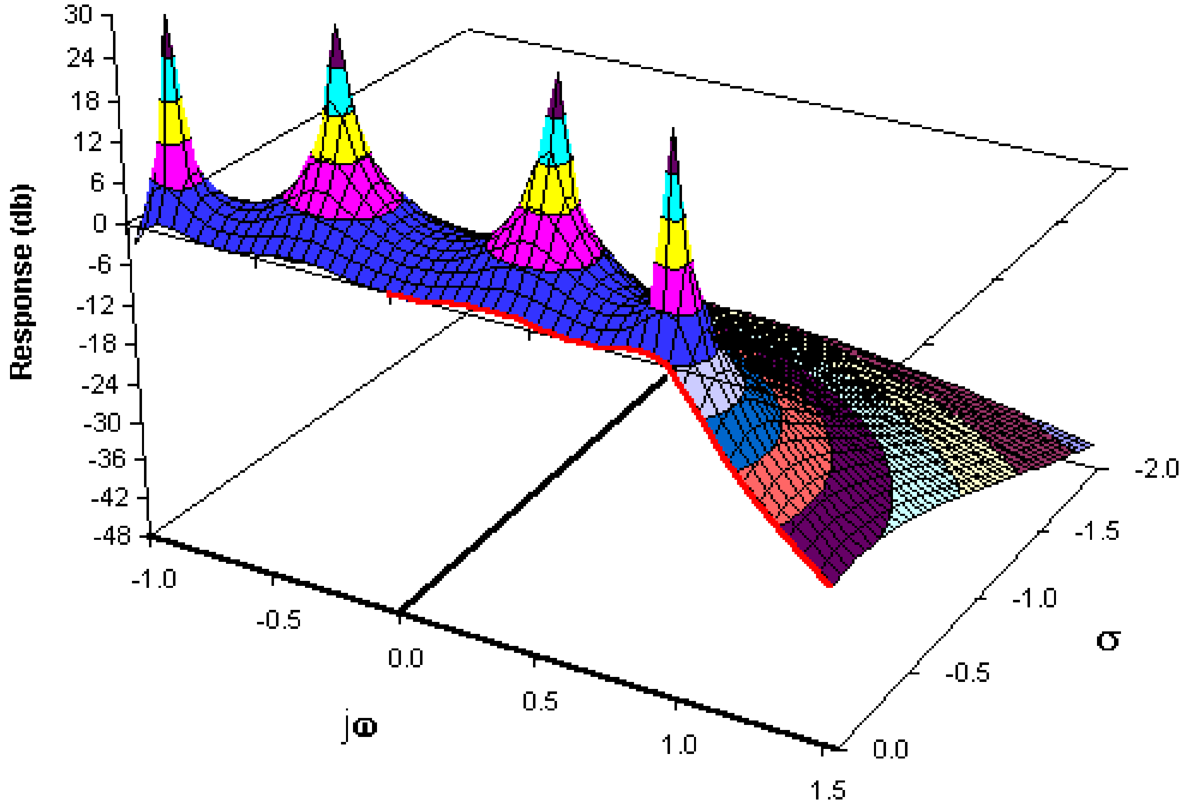
  $T_1(\omega) = \omega$ and $T_{n+1}(\omega) = 2\,\omega T_n(\omega) - T_{n-1}(\omega)$



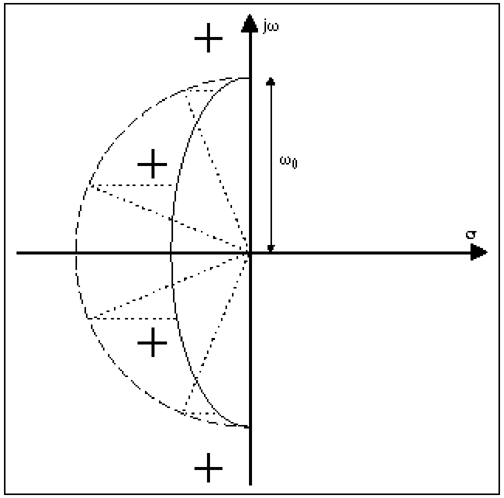




- Characteristics
  - Controlled equiripple in the pass-band
  - Sharper cutoff than Butterworth
  - Non-linear phase (Group delay distortion) ☹

# Chebyshev
## |H(s)| for n=4, r=1 (Type 1)



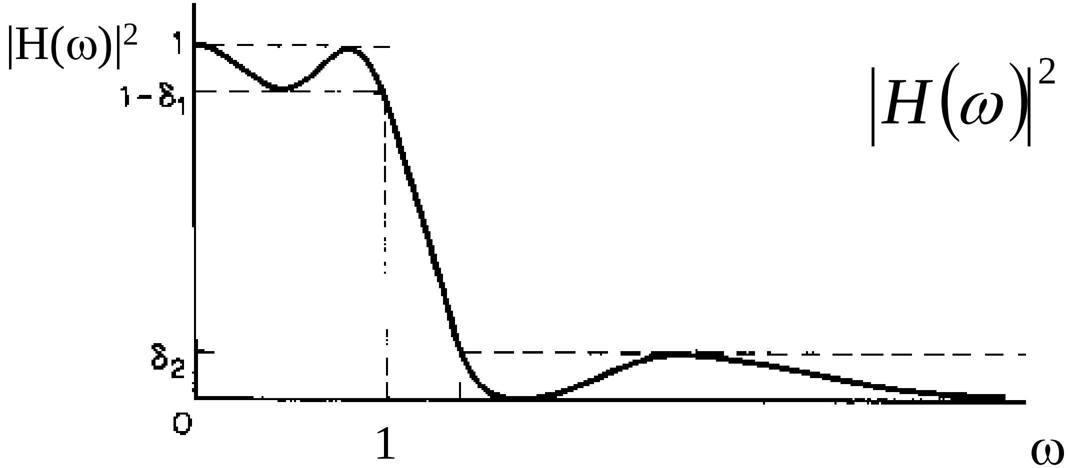

Poles lie on an ellipse




$$H(s) = 0.2457/( s^4 + 0.9528s^3 + 1.4539s^2 + 0.7426s + 0.2756)$$

# Elliptic Function

- $F(\omega) = U_n(\omega)$ – the Jacobian elliptic function

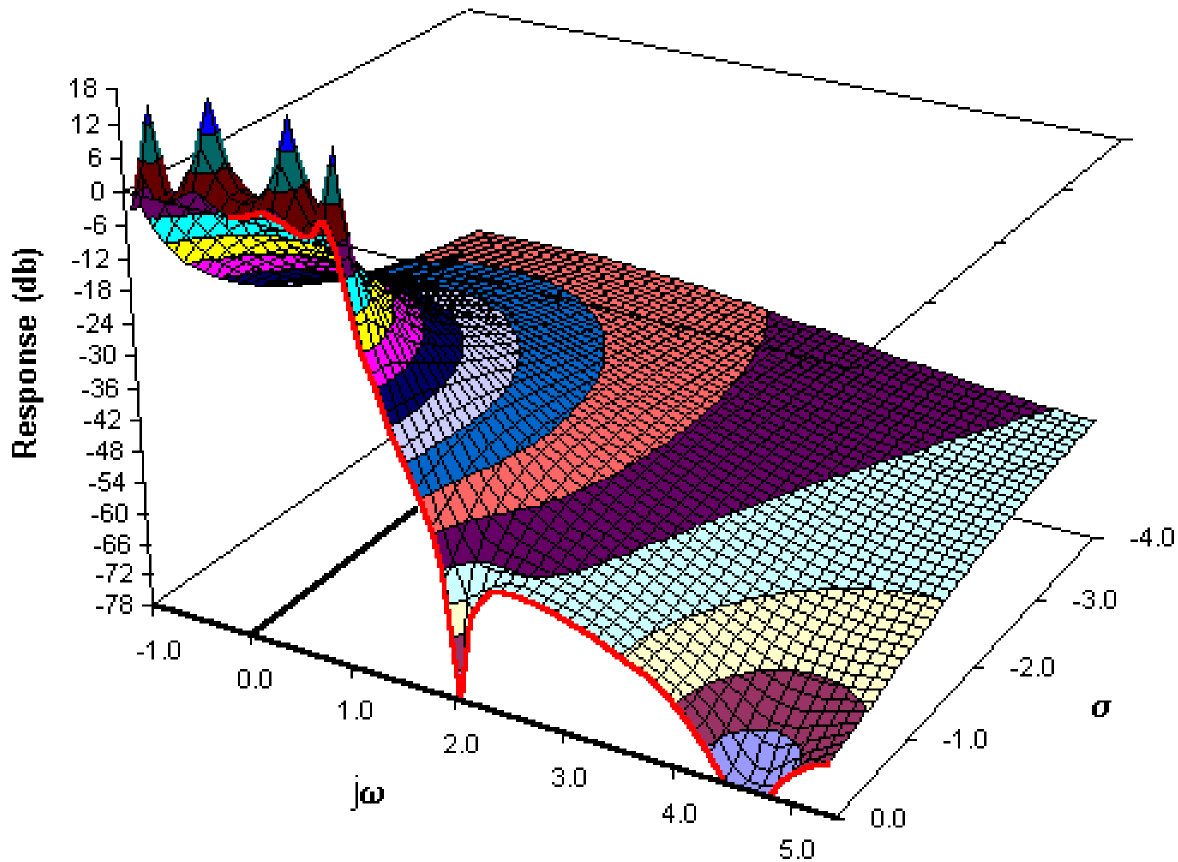


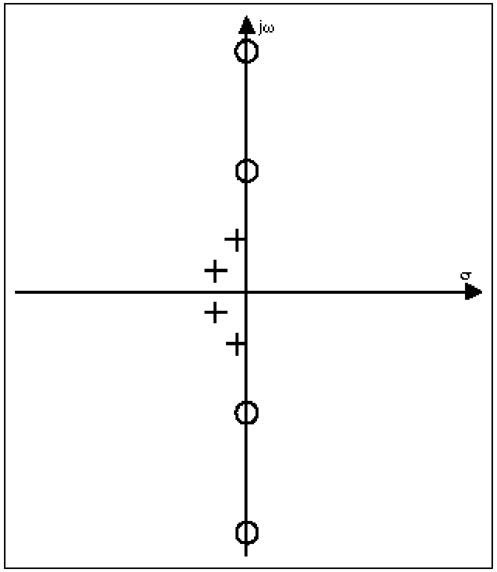

$$|H(\omega)|^2 = \frac{1}{1+\varepsilon^2 U_n^2(\omega)}$$

- S-Plane
  - Poles approximately on an ellipse
  - Zeros on the jω-axis
- Characteristics
  - Separately controlled equiripple in the pass-band and stop-band
  - Sharper cutoff than Chebyshev (optimal transition band)
  - Non-linear phase (Group delay distortion) ☹

# Elliptic Function

## H(s) for n=4, $r_p$=3, $r_s$=50

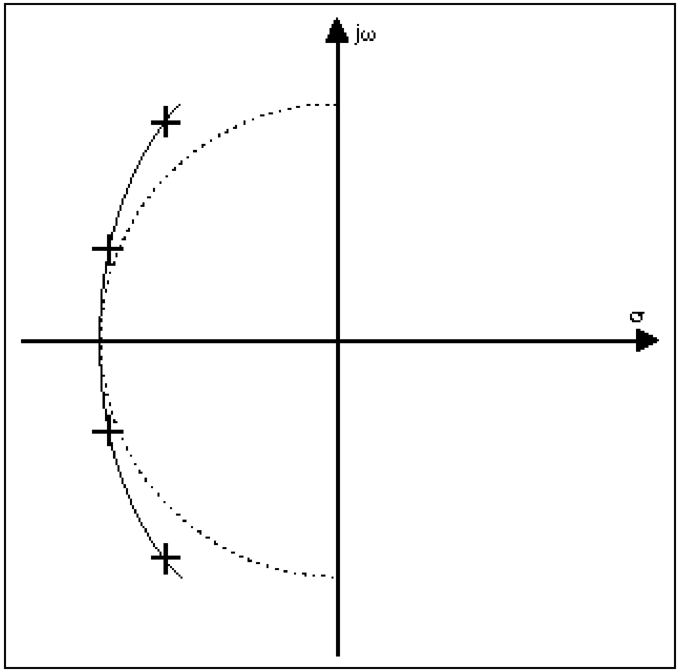



$$H(s) = (0.0032s^4 + 0.0595s^2 + 0.1554)/( s^4 + 0.5769s^3 + 1.2227s^2 + 0.4369s + 0.2195)$$

# Bessel Filter

- Butterworth and Chebyshev filters with sharp cutoffs (high order) carry a penalty that is evident from the positions of their poles in the s plane. Bringing the poles closer to the jω axis increases their Q, which degrades the filter's transient response. Overshoot or ringing at the response edges can result.
- The Bessel filter represents a trade-off in the opposite direction from the Butterworth. The Bessel's poles lie on a locus further from the jω axis. Transient response is improved, but at the expense of a less steep cutoff in the stop-band.



# Practical Filter Design

- Use a tool to establish a prototype design
  - MatLab is a great choice
  - See http://doctord.webhop.net/courses/Topics/Matlab/index.htm
    for a Matlab tutorial by Dr. Bouzid Aliane; Chapter 5 is on filter design.
- Check your design for ringing/overshoot.
  - If detrimental, increase the filter order and redesign to exceed the frequency response specifications
  - Move poles near the jω-axis to the left to reduce their Q
  - Check the resulting filter against your specifications
    - Moving poles to the left will reduce ringing/overshoot,
      but degrade the transition region.