# Response of LTI Systems Using Laplace Transforms



Where *h*(*t*) is an impulse response, is called the system function or transfer function and it completely characterizes the input/output relationship of an LTI system. We can use it to
determine time responses of LTI systems.

## Transfer Functions

We can use Laplace Transforms to solve differential equations for systems (assuming the system is initially at rest for one-sided systems) of the form:



Taking the Laplace Transform of both sides of this equation and using the Differentiation Property, we get:



From this, we can define the transfer function, ***H*(*s*)**, as:



Which is the ratio of two polynomials in “s”

## Partial Fraction Expansion

Instead of taking contour integrals to invert Laplace Transforms, we will use Partial Fraction Expansion. We review it here. Given a Laplace Transform,



If m isn’t less than n, perform **polynomial division** and then the remainder can be analyzed by Partial Fraction Expansion.

We write its Partial Fraction Expansion as:



where

Assuming that all of the poles have unique values!



is the *residue* of the pole at *pj*.

Thus



because the Inverse Laplace Transform of



### Convolution

An important property of Laplace Transforms is that the Laplace transform of the convolution of two signals is the product of their Laplace transforms:



This is useful for studying LTI systems. In fact, we can completely characterize an LTI system from:

1. The system differential equation or
2. the system transfer function *H*(*s*) or
3. the system impulse response *h*(*t*).

**Example 1** Find *y*(*t*) where the transfer function *H*(*s*) and the input *x*(*t*) are given. Use Partial Fraction Expansion to find the output *y*(*t*):



First find the Laplace transform of x(t) from a table of Laplace transforms: X(s) = 1/(s+3)

Now find the Laplace Transform of the output by multiplying H(s) by X(s)

Y(s) = (3s+1) / [ (s+3)\*(s+2)\*(s+3) ]

To do partial fractions we need to separate out the poles, but one of the poles is repeated so we need to find

Y(s) = A/(s+2) + B/(s+3) + C/(s+3)2

The third term is to account for the repeated root.

Multiplying both sides of the equation by (s+2) yields:

(s+2)\* Y(s) = (3s+1) / [ (s+3)\*(s+3) ] = A + (s+2) \* [ B/(s+3) + C/(s+3)2 ]

Letting s ---> -2 now yields

-5 / [ (1)\*(1) ] = A so **A = -5**

If we now repeat the process but multiply both sides by (s+3)2 we get:

(s+3)2\* Y(s) = (3s+1) / [ (s+2) ] = A\*(s+3)2 + B\*(s+3) + C

Now take the limit as s ---> -3 and

(-9 + 1) / (-1) = C so **C = 8**

there are two methods to find the remaining constant, B

1. Take 2the derivative of both sides of the original equation and then B can be isolated by multiplying both sides by (s+3) and taking the limit as s ---> -3. This is the method shown in most textbooks.
2. go back to the original equation setting A = 1 and C = 8. Then put the three terms over a common denominator and the extra terms should cancel out to leave the “B” term.

Using the second method here:

Y(s) = -5/(s+2) + 8/(s+3) + B/(s+3)

Y(s) = [-5(s+3)2 + 8(s+2) + B(s+2)(s+3)] / [(s+2)(s+3)2]

Y(s) = [-5s2 - 30s - 45- +8s +16 + B(s2 + 5s + 6)] / [(s+2)(s+3)2]

Y(s) = (B-5)s2 +(5B - 22)s + (6B-45) / [(s+2)(s+3)2]

But Y(s) = (3s+1) / [ (s+3)\*(s+2)\*(s+3) ] so

B-5 = 0 or **B = 5** from the s2 term but as a check from the s1 term:

5B-22 = 3  **OK** and from the s0 term

6B-29 = 1 **OK**

We Have the partial fraction expansion of:

**Y(s) = -5/(s+2) + 5/(s+3) + 8/(s+3)2**

And using the [Table of Laplace Transforms](https://quip.com/cYXhA1GEHAw9)

**y(t) = [-5\*exp(-2t) +5\*exp(-3t) +8t\*exp(-3t)]\*U(t)**

### Stability

We saw that a condition for bounded-input bounded-output stability was:



Let's look at stability from a system function standpoint.

Given a Laplace Transform *H*(*s*), we expand *H*(*s*) with Partial Fraction Expansion:



The corresponding impulse response is:



What happens to *h*(*t*) as *t* → ∞? For a system to be stable, its impulse response must not blow up as *t* → ∞.

