# Sampling and Reconstruction

Introduction

This section introduces the relationship between continuous signals/systems and their representation as discrete-time signals/systems that allows the direct use of digital computing technology. Sampling forms the bridge between these two representations of the world.

Analog System → Sampling → Discrete-Time System

Discrete-Time System → Reconstruction → Analog System
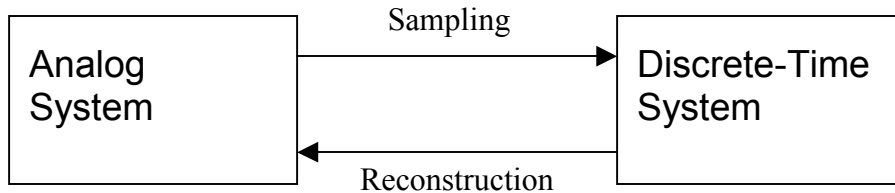
A Discrete-Time system has as its input and output sequences of values. Here it is assumed that these sequences represent samples of analog signals that are equally spaced in time (T_s seconds apart). Extensions to applications involving functions of other variables (e.g. distance, force, pressure. velocity, pixels, …) will be apparent.

$x(n*T_s)$ → Discrete-Time System → $y(n*T_s)$

x_n → Discrete-Time System → y_n
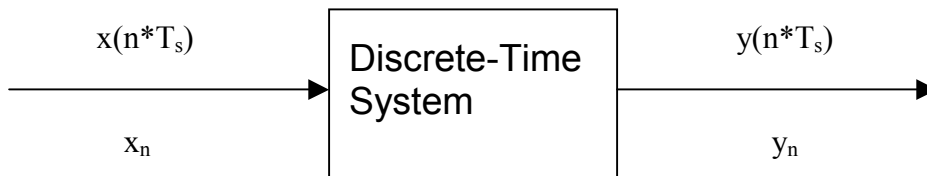
In order to develop the analysis tools for such a system, we must form an equivalent analog system.

$x(t)$ → Sample (T_s sec.) → $x(t, T_s)$ → Analog System → Re-Sample (T_s sec.) → Reconstruct → $y(t)$
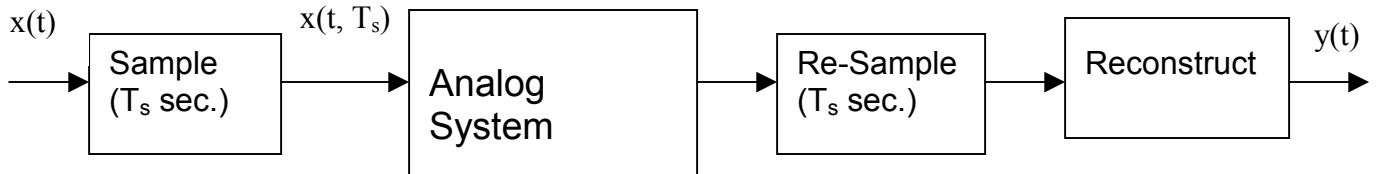
Where:

$$x_n = x(n * T_s) = x(t)\big|_{t=n*T_s}$$

and

$$y_n = y(n * T_s) = y(t)\big|_{t=n*T_s}$$

Analyzing this equivalent system using standard analog tools will establish a set of discrete-time tools including the Discrete Fourier Transform (DFT) and the z-Transform.

Another reason for examining the sampling/reconstruction process is that they form the actual hardware interface between analog and discrete-time systems. We will then be prepared to perform Digital Filtering of analog signals.

Sampling
Pulse Amplitude Modulation

PAM is where the analog signal after sampling takes on the form of a sequence of pulses T_s seconds apart. Each pulse carries information about the analog signal's amplitude at the time the pulse was generated.

There are two variants of PAM. The first, currently the dominant approach, called Natural Sampling is where the shape of each pulse is affected by the changing input during the pulse. The second is called Uniform Sampling (mainly used in the older laboratory grade samplers) where all of the pulses have the same shape, but different amplitudes. We will analyze both.

Natural Sampling in the Time Domain

Assume that we have a generalized, time-limited pulse centered at t = 0 as shown below.

p(t)
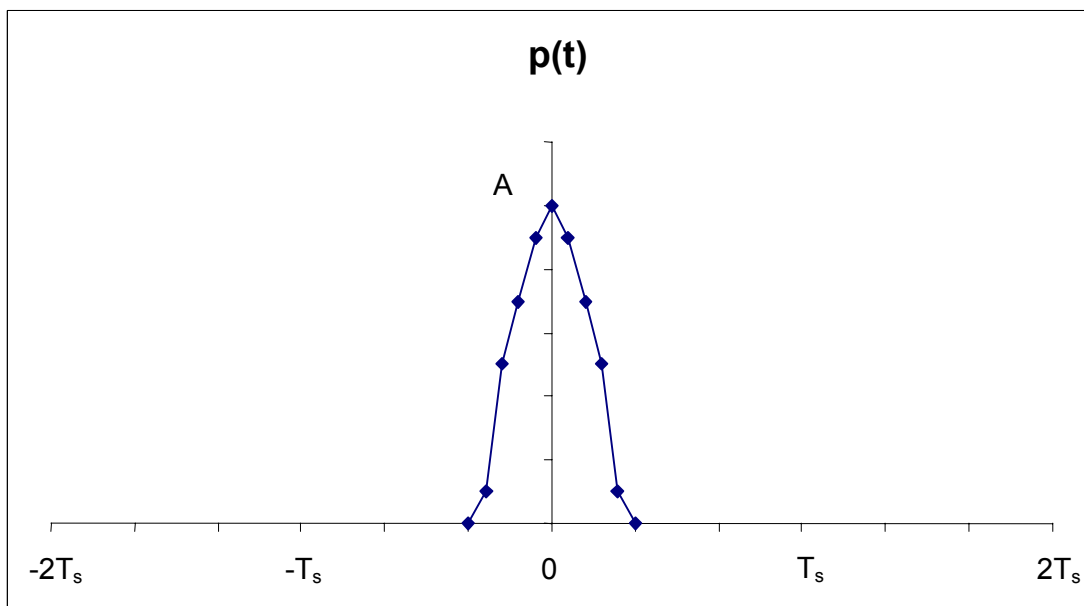
The Fourier Transform of this pulse is to denoted $P(j\omega)$. A periodic version of this pulse where the original pulse now repeats every T_s seconds is then:
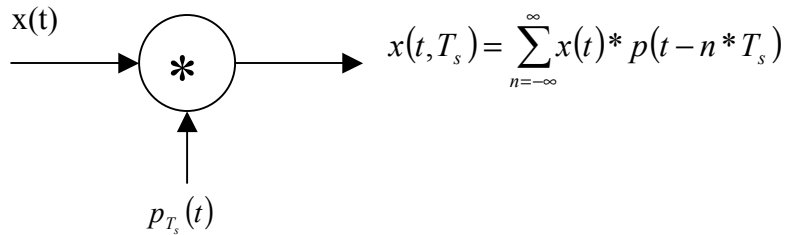
$p_T(t)$

A

-2T$_s$ -T$_s$ 0   T$_s$ 2T$_s$
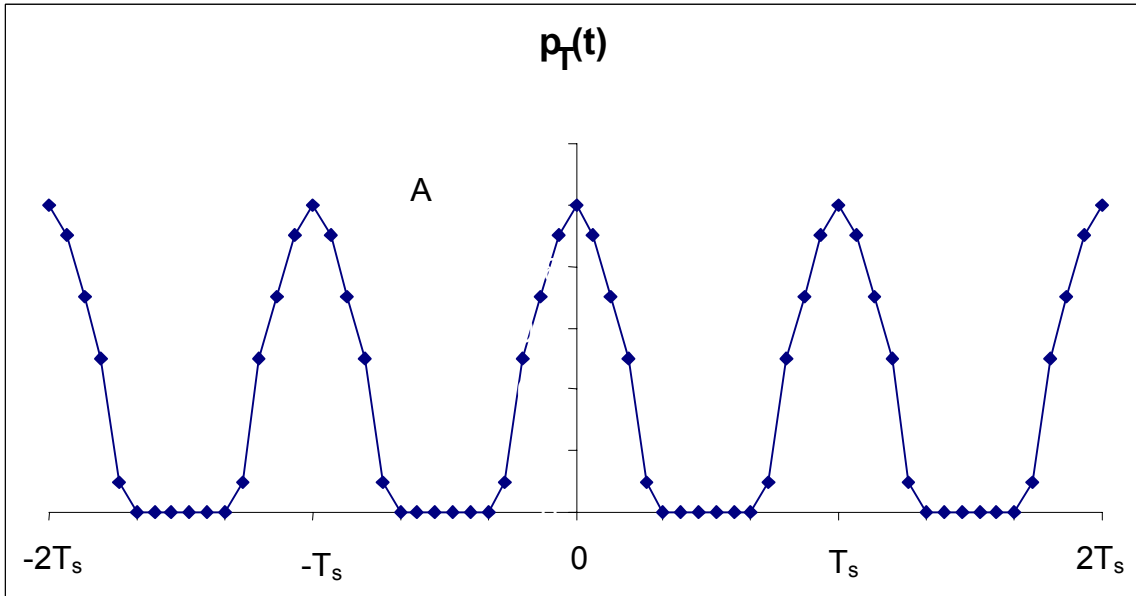
Note: $\lim_{T_s \to \infty} \lfloor p_{T_s}(t) \rfloor = p(t)$

$p_{T_s}(t)$, the pulse train, is periodic with period T$_s$ and we can express it as a sum of individual pulses.

$$p_{T_s}(t) = \sum_{n=-\infty}^{\infty} p(t - n * T_s)$$

"Natural Sampling" is then done by multiplying x(t), the input signal, by $p_{T_s}(t)$

x(t)

⊛

$x(t, T_s) = \sum_{n=-\infty}^{\infty} x(t) * p(t - n * T_s)$

$p_{T_s}(t)$

Uniform Sampling in the Time Domain

The second type of sampling, Uniform Sampling, results in a train of pulses in which each pulse has the same shape. Each pulse has a "strength" which is set by the value of the input at the sampling "instant". This method also goes by the name "Flat Top Sampling" when the transmitted pulses are rectangular. One way to implement uniform sampling is to precede a natural sampler with a "Sample and Hold" circuit.

x(t) → S & H → $x_{sh}(t)$ → $*$ → $x(t, T_s) = \sum_{n=-\infty}^{\infty} x(n * T_s) * p(t - n * T_s)$

$p_{T_s}(t)$
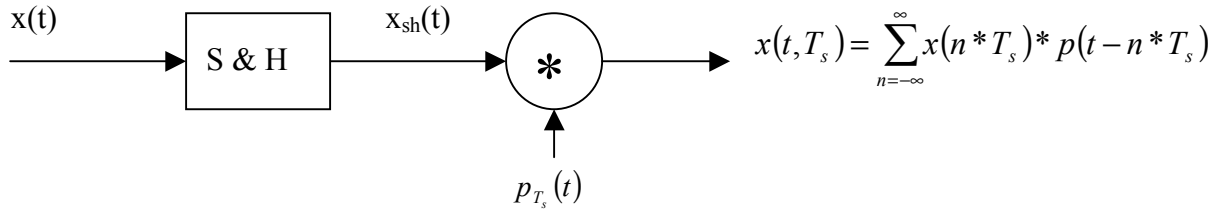
The Sample and Hold circuit has the following effect on an analog signal.
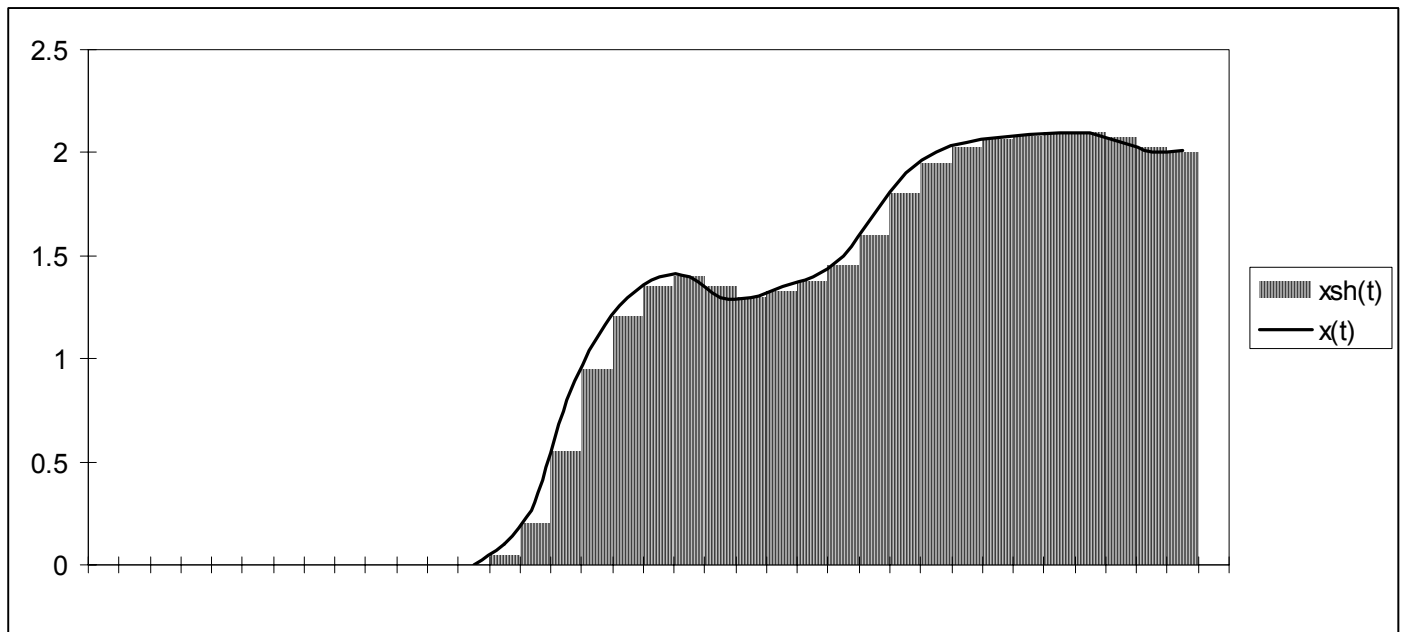
This is a non-linear distortion of the signal, but now the signal is constant during each pulse of the pulse train and multiplying leaves all of the resulting pulses having the same shape. A delay of ½ T_s must be introduced in the pulse train to center each pulse in a constant region of the modified signal. This has the effect of delaying the signal itself by $T_s/2$.

Frequency Domain Analysis of Sampled Signals

Introduction

Both types of sampling are practical. One, Natural Sampling, is simpler in the frequency domain while the other, Uniform Sampling, is simpler in the time domain. In the limiting case, as the pulse width approaches zero while holding the pulse area constant, both systems yield the same result. This limiting case is called Ideal (or Impulse) Sampling where each pulse is now the "Impulse" or "Dirac Delta" function, $\delta(t - n * T_s)$, and is the form of sampling described in many textbooks.

Natural Sampling in the Frequency Domain

Let P(jω) be the Fourier transform of p(t), the single sampling pulse of width Δt < T$_s$. The Fourier series for the periodic $p_{T_s}(t)$ is:
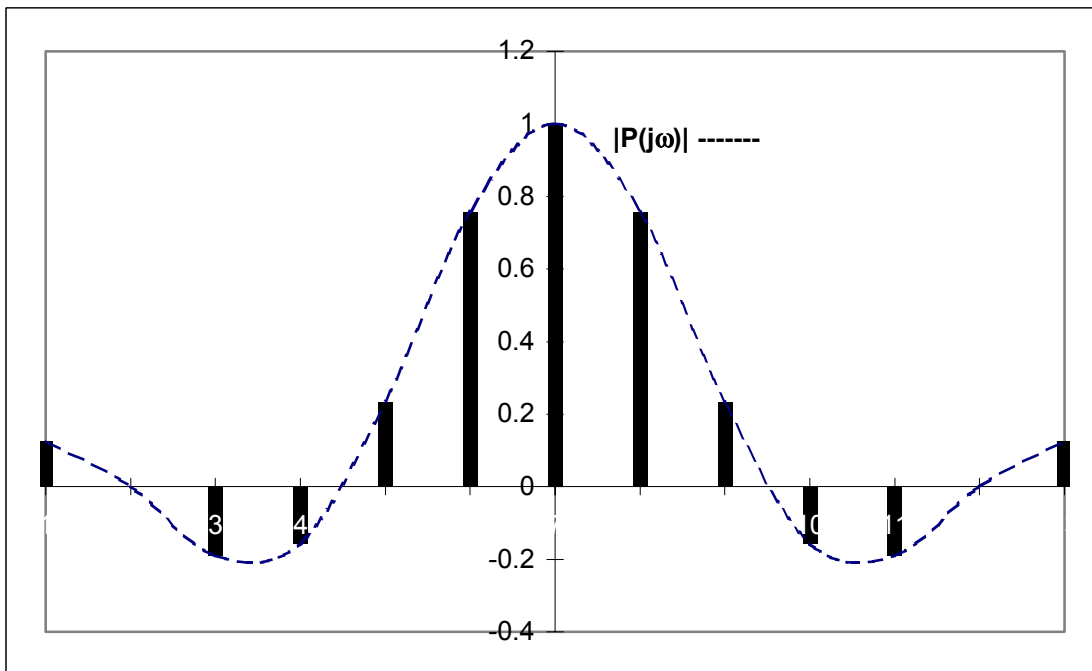
$$p_{T_s}(t) = \sum_{n=-\infty}^{\infty} P_n * \varepsilon^{jn\omega_s t} \quad \omega_s = \frac{2\pi}{T_s}, \text{ the sampling rate in radians per second.}$$

Where:

$$P_n = \frac{1}{T_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} p(t) * \varepsilon^{-jn\omega_s t} dt = \frac{1}{T_s} P(jn\omega_s) = \frac{1}{T_s} P\left(jn\frac{2\pi}{T_s}\right)$$

Which results in:

$$\Im\{p_{T_s}(t)\} = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} P\left(jn\frac{2\pi}{T_s}\right) * \delta\left(\omega - \frac{2\pi}{T_s}\right) = P_{T_s}(\omega)$$

Where the height of each "bar" denotes the strength of the delta function at that frequency.

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Since the Fourier Transform of a product of two time functions is the convolution of their individual Fourier Transforms. The Fourier Transform of the sampled version of the input, x(t), is:

$$X_{T_s}(j\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} P\left(jn\frac{2\pi}{T_s}\right) * X\left[j\left(\omega - \frac{2\pi}{T_s}\right)\right] = P_{T_s}(j\omega)$$
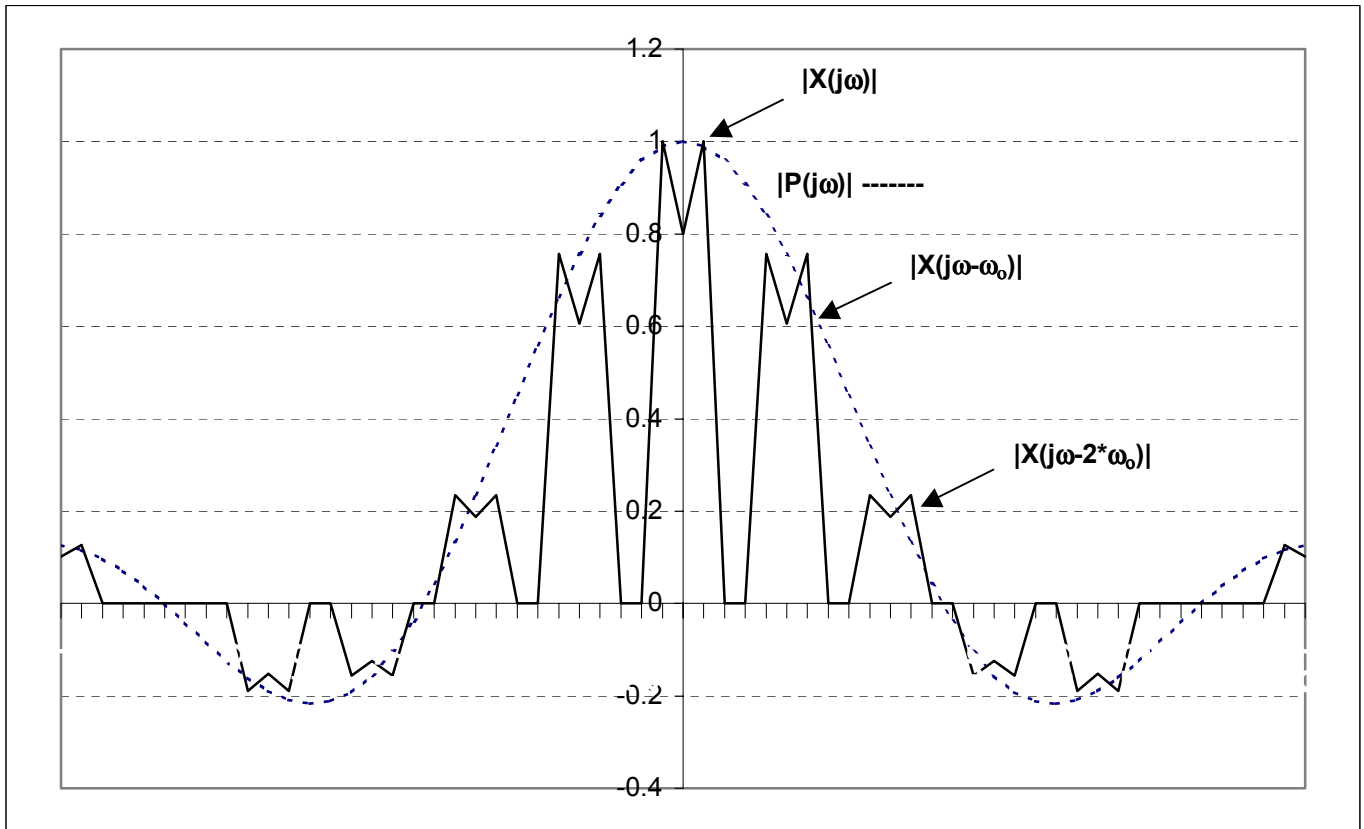
Notes:

1. Each repetition of $X(j\omega)$, the Fourier Transform of x(t), is identical. It only varies in magnitude according to the magnitude of $P(j\omega)$ at the center frequency of that repetition.

2. If ω_s (or f_s) increases (i.e. T_s decreases), the repetitions remain equally spaced in the frequency domain, but move further apart.

Aliasing

TBD

Uniform Sampling in the Frequency Domain

TBD