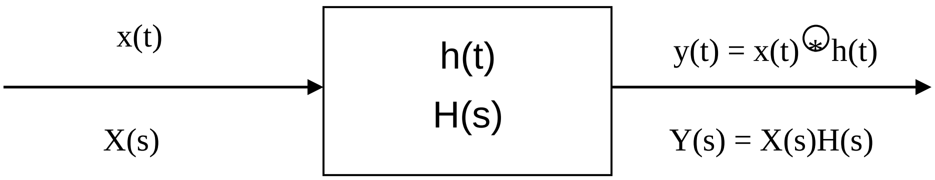
The Z-Transform

Introduction

A linear system can be represented in the complex frequency domain (s-domain where s = σ + jω) using the LaPlace Transform.



Where the direct transform is:

$$L\{x(t)\} = X(s) = \int_{t=0}^{\infty} x(t)\varepsilon^{-st}\,dt$$

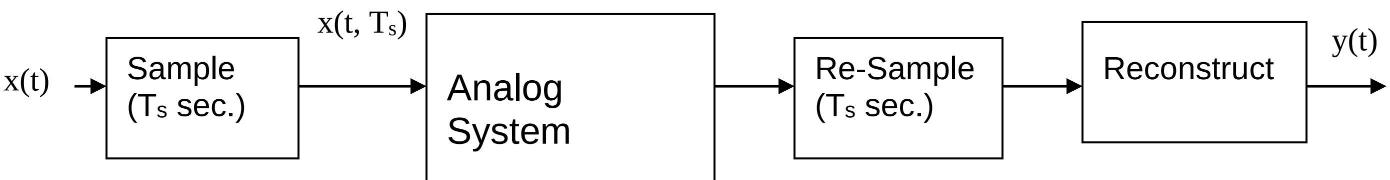
And x(t) is assumed zero for $t \leq 0$

The Inversion integral is a contour integral in the complex plane (seldom used, tables are used instead)

$$L^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j}\int_{s=\sigma-j\infty}^{\sigma+j\infty} X(s)\varepsilon^{st}\,ds$$

Where σ is chosen such that the contour integral converges.

If we now assume that x(t) is ideally sampled as in:



Where:

$$x_n = x(n*T_s) = x(t)\big|_{t=n*T_s}$$

and

$$y_n = y(n*T_s) = y(t)\big|_{t=n*T_s}$$

Analyzing this equivalent system using standard analog tools will establish the z-Transform.

Sampling

Substituting the Sampled version of x(t) into the definition of the LaPlace Transform we get

$$L\{x(t,T_s)\} = X_T(s) = \int_{t=0}^{\infty} x(t,T_s)\varepsilon^{-st}\,dt$$

But

$$x(t,T_s)=\sum_{n=0}^{\infty} x(t)*p(t-n*T_s)$$ (For x(t) = 0 when t < 0)

Therefore

$$X_T(s)=\int_{t=0}^{\infty}\left[\sum_{n=0}^{\infty} x(n*T_s)*\delta(t-n*T_s)\right]\varepsilon^{-st}dt$$

Now interchanging the order of integration and summation and using the sifting property of δ-functions

$$X_T(s)=\sum_{n=0}^{\infty} x(n*T_s)\int_{t=0}^{\infty}\delta(t-n*T_s)\varepsilon^{-st}dt$$

$$X_T(s)=\sum_{n=0}^{\infty} x(n*T_s)\varepsilon^{-nT_s s}$$ (We are assuming that the first sample occurs at t = 0+)

if we now adjust our nomenclature by letting:

$z = \varepsilon^{sT}$, $x(n*Ts) = x_n$, and $X(z)=X_T(s)\big|_{z=\varepsilon^{sT}}$

$$\boxed{X(z)=\sum_{n=0}^{\infty} x_n z^{-n}}$$

Which is the direct z-transform (one-sided; it assumes $x_n = 0$ for n < 0).

The inversion integral is:

$$\boxed{x_n=\frac{1}{2\pi j}\oint_c X(z)z^{n-1}dz}$$ (This is a contour integral in the complex z-plane)

(The use of this integral can be avoided as tables can be used to invert the transform.)

To prove that these form a transform pair we can substitute one into the other.

$$x_k=\frac{1}{2\pi j}\oint_c\left[\sum_{n=0}^{\infty} x_n z^{-n}\right]z^{k-1}dz$$

Now interchanging the order of summation and integration (valid if the contour followed stays in the region of convergence):

$$x_k=\frac{1}{2\pi j}\sum_{n=0}^{\infty} x_n\oint_c z^{k-n-1}dz$$

If "C" encloses the origin (that's where the pole is), the Cauchy Integral theorem says:

$$\oint_c z^{k-n-1}dz=\begin{matrix} o & \text{for } n\neq k \\ 2\pi j & \text{for } n=k\end{matrix}$$

And we get $x_k = x_k$ Q.E.D

An Example

Find the z-transform of

$\delta_{n-k} = {}^{o \quad if\ n \neq k}_{i \quad if\ n = k}$ This is the "Unit Pulse" at n = k (assume k > 0)

$$\Delta(z) = \sum_{n=0}^{\infty} \delta_{n-k} z^{-n}$$

$\boxed{\Delta(z) = z^{-k}}$ **(Note: dividing by z is equivalent to a delay of one sample time)**

A Short Table of z-Transforms

f(t) (sampled)	F(z)	Region of Convergence
U(t)	$\frac{z}{z-1}$	$\|z\| > 1$
δ_{n-k}	z^{-k}	$\|z\| > 1$
t	$\frac{Tz}{(z-1)^2}$	$\|z\| > 1$
t^2	$\frac{T^2 z(z+1)}{(z-1)^3}$	$\|z\| > 1$
ε^{at}	$\frac{z}{z-\varepsilon^{aT}}$	$\|z\| > \varepsilon^{at}$
$\sin(\beta t)$	$\frac{z * \sin(\beta T)}{z^2 - 2z * \cos(\beta T) + 1}$	$\|z\| > 1$
$\cos(\beta t)$	$\frac{z * [z - \cos(\beta T)]}{z^2 - 2z * \cos(\beta T) + 1}$	$\|z\| > 1$

Properties of the z-Transform

The z-transform has properties that are analogous to those of the LaPlace Transform. The following table has some of the more useful ones listed.

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad \Leftrightarrow \quad x[n] \equiv \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

where C is a closed contour that includes z=0

	Signal		z-Transform
	$x[n]$	$\Leftrightarrow$	$X(z)$
Superposition	$ax[n]+by[n]$	$\Leftrightarrow$	$aX(z)+bY(z)$
Time Shifting	$x[n-n_0]$	$\Leftrightarrow$	$z^{-n_0}X(z)$
	$e^{j\Omega_0 n}x[n]$	$\Leftrightarrow$	$X\left(e^{-j\Omega_0}z\right)$
	$z_0^n x[n]$	$\Leftrightarrow$	$X\left(\frac{z}{z_0}\right)$
	$a^n x[n]$	$\Leftrightarrow$	$X\left(a^{-1}z\right)$
Time inversion	$x[-n]$	$\Leftrightarrow$	$X\left(z^{-1}\right)$
Time Convolution	$x[n]*y[n]$	(convolution)	$X(z)Y(z)$
Frequency Differentiation	$n\,x[n]$	$\Leftrightarrow$	$-z\frac{dX(z)}{dz}$
Summation	$\sum_{k=-\infty}^{n} x[k]$	$\Leftrightarrow$	$\frac{1}{1-z^{-1}}X(z)$

You should familiarize yourself with these as they will be used, along with the table of transforms to move between time series and the z-domain.

Finding the Inverse z-Transform

There are three common ways to find the time series, x_n when X(z) is given:

1. Infinite Series – done by dividing out the rational polynomial in z
2. Partial Fraction Expansion – Same as in LaPlace
3. The Inversion Integral – a contour integral in the complex z-plane

Example: $F(z)=\dfrac{2z}{(z-2)(z-1)^2}$, determine f_n

A. By Infinite Series

$$F(z)=\frac{2z}{z^3-4z^2+5z-2}$$

Now divide (long division) with the polynomials written in descending powers of z

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                    2z^-2+8z^-3+22z^-4+52z^-5+114z^-6+…
Z^3-4z^2+5z-2|2z
                    2z-8+10z^-1-4z^-2
                        8-10z^-1+04z^-2
                        8-32z^-1+40z^-2-16z^-3
                           22z^-1-36z^-2+016z^-3
                           22z^-1-88z^-2+110z^-3-44z^-4
                                   52z^-2-094z^-3+044z^-4
                                   52z^-2-208z^-3+260z^-4-104z^-5
                                          114z^-3-216z^-4+104z^-5
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$$\therefore \quad F(z)=\sum_{n=0}^{\infty} f_n z^{-n} = 2z^{-2}+8z^{-3}+22z^{-4}+52z^{-5}+114z^{-6}+\dots$$

And the time sequence for f_n is

n	0	1	2	3	4	5	6	…
f_n	0	0	2	8	22	52	114	…

Note that this method does NOT give a closed form for the answer, but it is a good method for finding the first few sample values or to check out that the closed form given by another method at least starts out correctly.

B. By Partial Fraction Expansion

$$F(z)=\frac{2z}{(z-2)(z-1)^2}=\frac{k_1 z}{z-2}+\frac{k_2 z}{z-1}+\frac{k_3 z}{(z-1)^2}$$

To find k1 multiply both sides of the equation by (z-2), divide by z, and let z→2

$$\frac{2z}{(z-1)^2}=k_1 z+\frac{k_2 z(z-2)}{z-1}+\frac{k_3 z(z-2)}{(z-1)^2}$$

$$\frac{2}{(z-1)^2}=k_1+\frac{k_2(z-2)}{z-1}+\frac{k_3(z-2)}{(z-1)^2}$$

$$\left.\frac{2}{(z-1)^2}\right|_{z=2}=k_1+\left.\frac{k_2(z-2)}{z-1}\right|_{z=2}+\left.\frac{k_3(z-2)}{(z-1)^2}\right|_{z=2}$$

or

k_1 = 2

Similarly to find k_3 multiply both sides by $(z-1)^2$, divide by z, and let z→1

$$\frac{2}{(z-2)}=\frac{k_1(z-1)^2}{z-2}+k_2(z-1)+k_3 z$$ Equation A

And

k_3 = -2

Finding k2 requires going back to Equation A above and taking the derivative of both sides

$$\frac{2}{(z-2)}=\frac{k_1(z-1)^2}{z-2}+k_2(z-1)+k_3 z$$

$$-\frac{2}{(z-2)^2}=k_1\left[\frac{2(z-1)}{z-2}-\frac{2(z-1)^2}{(z-2)^2}\right]+k_2$$

Now again let z→1

k_2 = -2

$$\therefore\quad F(z)=\frac{2z}{z-2}-\frac{2z}{z-1}-\frac{2z}{(z-1)^2}$$

C. Using the Inversion Integral

TBD

H.W. Find the inverse z-Transform of

$$F(z)=\frac{z(z^2-2z-1)}{(z^2+1)^2}$$

We can check the answer by putting the three terms over the common denominator

$$F(z) = 2z\frac{(z-1)^2 - (z-1)(z-2) - (z-2)}{(z-1)(z-2)^2}$$

$$F(z) = 2z\frac{z^2 - 2z + 1 - z^2 + 3z - 2 - z + 2}{(z-1)(z-2)^2}$$

$$F(z) = 2z\frac{1}{(z-1)(z-2)^2}$$ It checks out!

